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ABSTRACT

This is unit thirteen of a fifteen-unit SMSG secondary school text for high school students. The text is devoted almost entirely to mathematical concepts which all citizens should know in order to function satisfactorily in our society. Chapter topics include approximations, solution sets of mathematical sentences, and quadratic functions. (MP)

SCHOOL MATHEMATICS STUDY GROUP

UNIT NUMBER THIRTEEN

Chapter 22. Approximations

Chapter 23. Solution Sets of Mathematical Sentences

Chapter 24. Quadratic Functions

TABLE OF CONTENTS

Chapter 22

APPROXIMATIONS

22-1. A Famous Problem

What is the shape of the most beautiful rectangle? According to the ancient Greeks, the rectangle ABCD shown here has the most beautiful shape if, when $\overline{EF}$ is drawn so that AEFD is a square, the remaining rectangle BCFE is similar to the original rectangle ABCD.

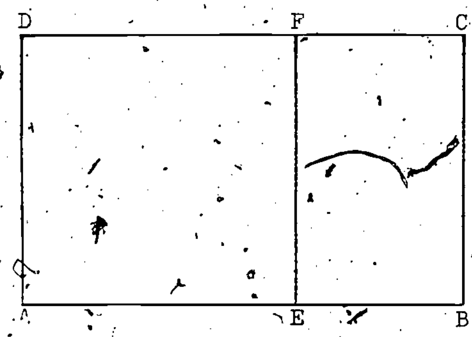


Figure 1a

Let us see what this means in terms of numbers. We choose the unit of length so that the altitude of ABCD is 1. Let EB = x. If BCFE and ABCD are similar, then $\frac{EB}{BC} = \frac{AD}{AB}$. That is,

$$\frac{x}{1} = \frac{1}{1 + x}.$$

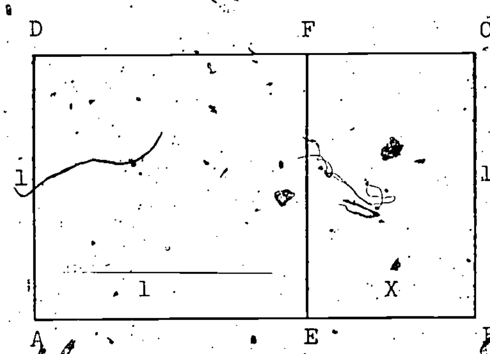


Figure 1b

This result may be more apparent if we draw the rectangles ABCD and BCFE in a different position.

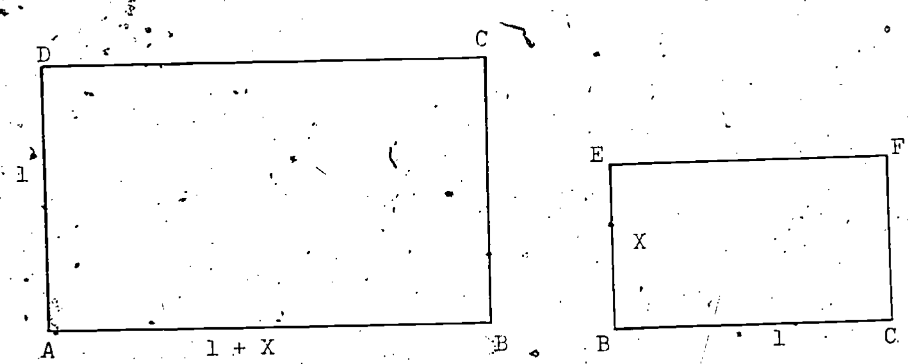


Figure 1c

Therefore the Greek requirement amounts to saying that the most beautiful rectangle (often called the golden rectangle) is one for which

$$x = \frac{1}{1 + x}.$$

The figure shows the graphs of $y = x$ and $y = \frac{1}{1 + x}$. The graph of $y = x$ is familiar. To graph $y = \frac{1}{1 + x}$ we locate the points $(0, 1)$, $(\frac{1}{2}, \frac{2}{3})$ and $(1, \frac{1}{2})$ and notice that $\frac{1}{1 + x}$ decreases as x gets larger.

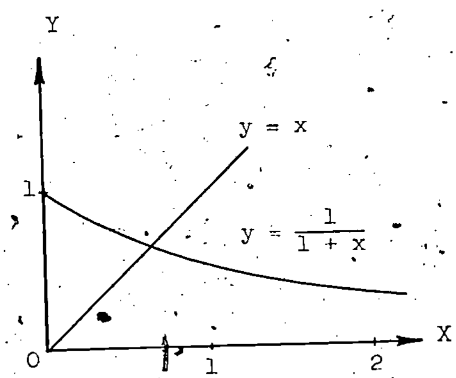


Figure 2

The two graphs intersect at a point P whose x-coordinate is between $\frac{1}{2}$ and 1.

Since P lies on both graphs, the value of y given by $y = x$ must be the same as the value of y given by $y = \frac{1}{1 + x}$. Therefore at P

$$x = \frac{1}{1 + x}.$$

We wish to find x so that this is true.

The straight line is above the curve to the right of the intersection P and below the curve to the left of P.

For the function

$$f : x \to \frac{1}{1 + x}$$

we wish the output $(\frac{1}{1 + x})$ to equal the input x. If we start with $x = 1$, we obtain the output $\frac{1}{2}$, which of course is not equal to 1. But suppose that we feed back the output $\frac{1}{2}$ as a new input. This gives $\frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$ as a new output. This result is fed back into the function as an input and so on. The following table shows the successive results.

Notice that the inputs are alternately greater and less than the corresponding outputs, as the inequality signs show.

Moreover, the approximations are getting closer together. Since $\frac{5}{8} = .625$ and $\frac{8}{13} = .61538 \ldots$ we have squeezed the required answer into an interval less than .01 in length.

Input x		Output $\frac{1}{1 + x}$
1	$>$	$\frac{1}{2}$
$\frac{1}{2}$	$<$	$\frac{2}{3}$
$\frac{2}{3}$	$>$	$\frac{3}{5}$
$\frac{3}{5}$	$<$	$\frac{5}{8}$
$\frac{5}{8}$	$>$	$\frac{8}{13}$

Table 1

But before carrying the approximation further it is interesting to notice that $\frac{3}{5}$ and $\frac{5}{8}$ are both reasonably close to the number that we require.

So, if the Greeks were right, either a 3×5 card or a 5×8 card should be very pleasing to the eye. It is perhaps no accident that cards of these sizes are in fact very popular.

Exercises 22-1a

(Class Discussion)

On a piece of graph paper, draw a rectangle 8 units long and 5 units wide. (Choose your unit so that the rectangle is as large as possible.) Show the 3×5 rectangle which remains when a square 5 units on a side is marked off. Draw a second 5×8 rectangle with a 3×5 rectangle in one corner. Are these rectangles similar? You can test similarity by determining whether two diagonals are collinear. Explain.

We shall now show the "feed-back" method on the graph, starting with the input 1. The output is $AB = \frac{1}{2}$, the ordinate of the point B on the graph of $y = \frac{1}{1 + x}$. We wish to use $\frac{1}{2}$ as the next input. We choose $OC = AB = \frac{1}{2}$. Then $CD = \frac{2}{3}$ is the new output.

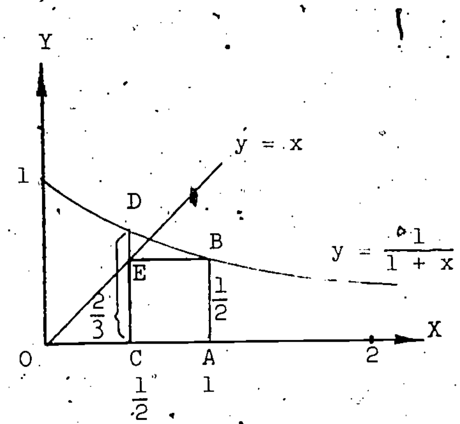


Figure 3

But notice that $\overline{CD}$ crosses the line $y = x$ at a point E which is on the same level as B. We might have expected this because if $y = x$, $CE = OC$ and we chose $OC = AB$.

Let us draw a new figure to bring out the facts more clearly. Start at A, go up to B on $y = \frac{1}{1 + x}$, across to E on $y = x$ and up to D on $y = \frac{1}{1 + x}$. To continue we must feed in $CD = EH = OF$. This gives us FG.

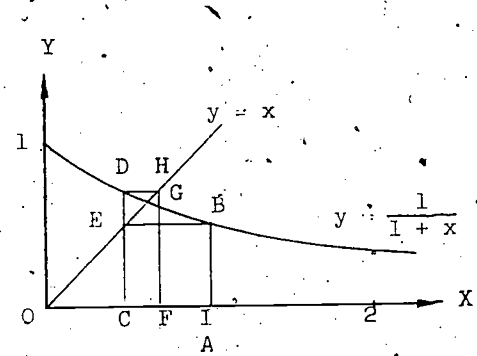


Figure 4

We show the essentials in Figure 5 where we can arrive at G by the path ABEDHG.

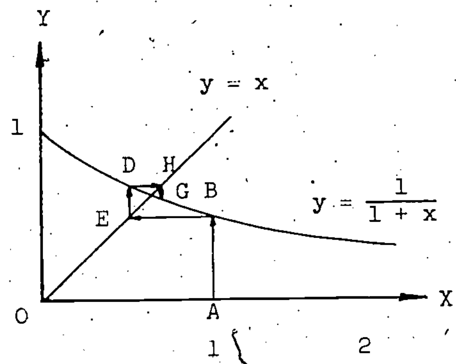


Figure 5

Exercises 22-1b

1. Measure some paperback books and divide the width of the front cover by the height of the front cover to see what numbers you get. (Reduce to decimals and compare with the numbers which we have been calculating.)

2. Continue the "feedback" process started in the text where each new choice of x is the previous value of $\frac{1}{1 + x}$. Start with $x = \frac{8}{13}$ and write the next three approximations in fractional form. Write the last two approximations just obtained in decimal form (carry the results to at least 4 decimal places).

3. By listing the fractions in Table 1 that approximate x and also the other fractions found in Exercise 2, see if you can continue the list to include five more items simply by observing the pattern.

4. Suppose that for some input a, in the function $f : x \rightarrow \frac{1}{1 + x}$,

$$a < \frac{1}{1 + a}.$$

Let b represent the output $\frac{1}{1 + a}$.

Show that if next time we use b as the input, then

$$b > \frac{1}{1 + b}.$$

That is, if a is too small, then b is too large.

Hint: Start with $a < b$, and add 1 to both sides of the inequality.

Prove that

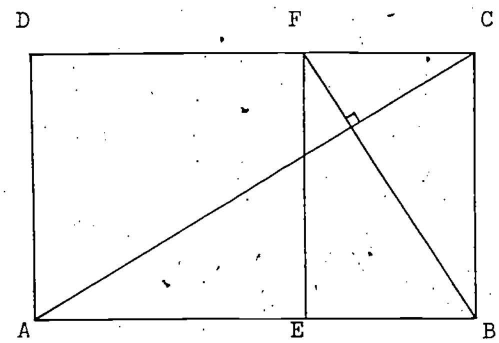
$$\frac{1}{1 + a} > \frac{1}{1 + b}$$

and therefore

$$b > \frac{1}{1 + b} .$$

5. Repeat the proof in Exercise 4 after interchanging $>$ and $<$.

6. Show that a diagonal of the golden rectangle must be perpendicular to a diagonal of the smaller rectangle that remains when a square is cut off.



7. Copy Figure 5 in this section on graph paper using a large scale and drawing the graph of $y = \frac{1}{1 + x}$ with great care. Continue the construction as far as practicable. What seems to happen?

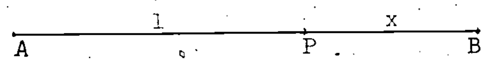
8. In this picture of the Parthenon, measure the dimensions of the rectangle shown. Divide the longer dimension by the shorter and compare the result with 1.62.

22-2. The Golden Mean

If we continue the process started in the previous section, it turns out that to nine decimal places

$$x \doteq .618033989.$$

If a line segment $\overline{AB}$



is divided by a point P into two parts so that $\frac{PB}{AP} = \frac{AP}{AB}$, $\frac{PB}{AP}$ is called the golden mean. Hence, if $AP = 1$ is the unit of length and $PB = x$, then

$$\frac{x}{1} = \frac{1}{1 + x}.$$

This equation has an approximate solution $.618\ldots$ as we have seen.

We may well ask whether this solution is rational or irrational. In other words, does there exist any choice of integers p and q so that

$$\frac{p}{q} = \frac{1}{1 + \frac{p}{q}},$$

that is, so that

$$\frac{p}{q} = \frac{q}{p + q}? \tag{1}$$

We use the same method that was used to prove the irrationality of $\sqrt{2}$.

If there is a solution of (1), then the fraction $\frac{p}{q}$ can be written in lowest terms. If this is done and if $\frac{q}{p + q}$ is equal to $\frac{p}{q}$, it must be true that the denominator $p + q$ is a multiple of q. That is,

$$p + q = q, \text{ or } 2q, \text{ or } 3q, \text{ or} \ldots$$

and

$$p = 0, \text{ or } q, \text{ or } 2q, \text{ or} \ldots$$

Finally

$$\frac{p}{q} = 0, \text{ or } 1, \text{ or } 2, \text{ or} \ldots$$

Since we assumed that $x = \frac{p}{q}$ and [illegible] $0 < x < 1$, there is no fraction $\frac{p}{q}$ which can be equal to x. That is, x is irrational.

As you know, this means that the decimal representation of x is non-terminating and non-repeating.

Exercises 22-2

1. In the list of approximations $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \ldots$ discover a rule which enables you to write any fraction from the preceding one in the list.

2. Follow the method of this section to show that there is no rational number $x = \frac{p}{q}$ for which $x = \frac{1}{x + 2}$.

3. Show that

$$x = \frac{c}{x + 1}$$

has a rational solution only if $c = n(n + 1)$. Note that $n(n + 1)$ represents the product of two successive integers n and $n + 1$. In this case, n is a solution of $x = \frac{c}{x + 1}$. Hint: Show that if $\frac{p}{q}$ is a solution of $x = \frac{c}{x + 1}$ it must be true that $\frac{p}{q} = 0$ or 1 or 2 or $\ldots$, that is, $\frac{p}{q}$ must be an integer n. In this case, c must be $n(n + 1)$.

22-3. The Fibonacci Numbers

Let us return to the fractions which we have used in the attempt to solve the equation

$$x = \frac{1}{1 + x}.$$

Written in succession these fractions are $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \ldots$.

Notice that

(1) The numerator of each fraction (after the first) is the same as the denominator of the previous fraction.

(2) The denominator of each fraction (after the first) is the sum of the numerator and denominator of the previous fraction.

More briefly, if $\frac{a}{b}$ is one fraction in the list, the next one is $\frac{b}{a + b}$.

The list of numbers used for the numerators and denominators, that is,

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

is a famous one. Each number (after the first two) is the sum of the preceding two numbers. These numbers are called the Fibonacci numbers after Leonardo Fibonacci (1170-1250) a great mathematician also called Leonardo of Pisa. Fibonacci means "Son of Good Fortune". The name is pronounced fib-on-ah'-chee.

Leonardo popularized the Hindu-Arabic decimal system in a book, Il Liber Abaci published in 1202. He introduced the Fibonacci series in a problem about the breeding of rabbits.

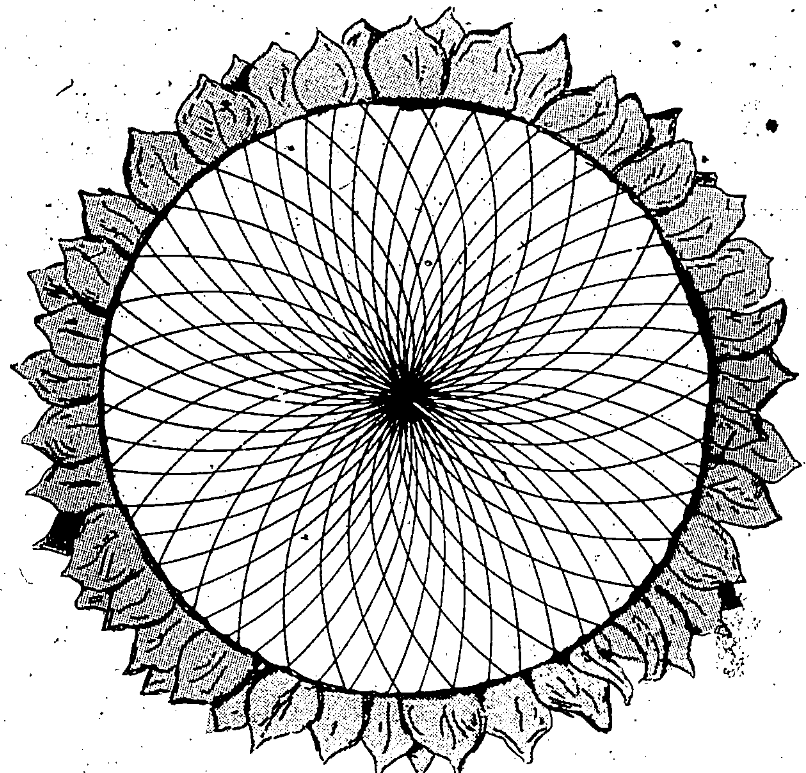
The Rabbit Problem

A pair of rabbits is placed in a walled enclosure to find out how many offspring this pair will produce in the course of a year if each pair of rabbits gives birth to a new pair each month starting with the second month of its life. Since the first pair has offspring in the first month, double the number, and in this month there are two pairs. Of these, one pair, the first, gives birth in the following month as well, so that in the second month there are three pairs. Of these, two pairs have offspring in the following month, so that in the third month two additional pairs of rabbits are born, and the total number of pairs of rabbits in this month reaches five. Three of these five pairs have offspring that month, and the number of pairs reaches eight in the fourth month. Five of these pairs produce another five pairs, which, together with the 8 pairs already in existence, make 13 pairs in the fifth month. Five of these 13 pairs have no offspring that month, while the remaining eight pairs do give birth, so that in the sixth month there are 21 pairs.*

Pairs:
1
first month:
2
second month:
3
third month:
5
fourth month:
8
fifth month:
13
sixth month:
21
seventh month:
34
eighth month:
55
ninth month:
89
tenth month:
144
eleventh month:
233
twelfth month:
377

* The Fibonacci Numbers, N. N. Vorobyov, D. C. Heath.

These numbers occur in very surprising places. For example, the seeds on most sunflower heads are arranged in 34 curves which cross 55 curves. Small heads have 21 curves intersecting 34 curves and large ones have 55 curves crossing 89 curves. The numbers 21, 34, 55 and 89 are Fibonacci numbers, as you can see.



There are many interesting relations among the Fibonacci numbers and extensive studies have been made of these relations. We shall study two of them which will be useful to us later.

We write the list once again for convenience.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Notice that the numbers increase in size (after the second 1). That is, if a and b are two successive Fibonacci numbers,

$$b \geq a.$$

(The equality sign applies only when $a = 1$ and $b = 1$.)

Instead of comparing successive numbers in the list, let us compare alternate numbers, for example 3 with 1, 5 with 2, 8 with 3 and so on. You will notice that in each case the larger number is more than twice the smaller number. Is this always true? Let us see. Suppose that a, b, and c are any three successive Fibonacci numbers.

$$\ldots a,\ b,\ c, \ldots$$

We know from the way the list is made that

$$c = a + b.$$

We also know that unless $a = 1$ and $b = 1$

$$b > a.$$

So we conclude that

$$c > a + a = 2a$$

unless $a = 1$ when of course $c = 2a$.

We have our first relation:

<u>If a, b, c, are three successive Fibonacci numbers, then</u>

$$c \geq 2a.$$

Returning to our list

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

let us square one of these numbers, say 5, and compare the result with the product of its neighbors (3 and 8).

$$5^2 = 25$$

$$3 \times 8 = 24.$$

The results differ by 1.

$$5^2 - (3 \times 8) = 1.$$

Similarly $8^2 = 64$ differs from $5 \times 13 = 65$ by 1. Notice however that to get 1 this time we must subtract in the opposite order.

$$(5 \times 13) - 8^2 = 1.$$

You can test this relation for yourself. It always seems to work. Can we prove that no matter how far out in our list we go, the difference in the proper order is equal to 1? Let us see.

We must see what happens when we progress from one set of three successive Fibonacci numbers to the next set, for example from 5, 8, 13

to 8, 13, 21, We have found that

$$(5 \times 13) - 8^2 = 1.$$

Now we ask about

$$(8 \times 21) - 13^2.$$

Of course we can easily show that the result is $168 - 169 = -1$. However, it is more useful to connect $(8 \times 21) - 13^2$ with $(5 \times 13) - 8^2$. We do this as follows:

$$\begin{aligned}
&(8 \times 21) - 13^2 \\
&= 8 \times (8 + 13) - 13 \times 13 \\
&= 8 \times (8 + 13) + (-13) \times (8 + 5) \\
&= (8 \times 8) + (8 \times 13) - (13 \times 8) - (13 \times 5) \\
&= 8^2 - (13 \times 5),
\end{aligned}$$

which is the opposite of $(5 \times 13) - 8^2$. It follows that

$$(8 \times 21) - 13^2 = -1.$$

Let us generalize this method. Suppose that a, b, c, d are any four successive Fibonacci numbers and that we know that

$$ac - b^2 = 1 \text{ or } -1.$$

What about

$$bd - c^2?$$

We use the fact that $d = b + c$ and $c = b + a$ to write

$$\begin{aligned}
bd - c^2 &= b\cdot(b + c) - c \cdot c \\
&= b\cdot(b + c) + (-c)(b + a) \\
&= b^2 + bc - cb - ca \\
&= b^2 - ca, \\
&= -(ac - b^2).
\end{aligned}$$

Therefore if $ac - b^2 = 1$, $bd - c^2 = -1$ and if $ac - b^2 = -1$, $bd - c^2 = 1$.

Exercises 22-3

1. Continue the list of Fibonacci numbers until you have 15 of them.
2. The Lucas numbers are formed if we start with 1 and 3 instead of 1 and 1, and obtain any later number by adding the previous

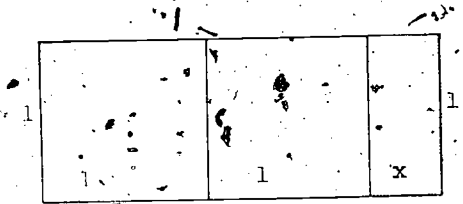
two. Thus we have $1, 3, 4, 7, \ldots$

(a) List the first 15 Lucas numbers.

(b) Write the list of Lucas numbers under the list of Fibonacci numbers and verify that the ninth Lucas number is the sum of the eighth and tenth Fibonacci numbers.

(c) Verify from your lists that the n^{th} Lucas number is the sum of the $(n - 1)$st and $(n + 1)$st Fibonacci numbers.

3. A rectangle is to be formed so that if we cut off 2 squares, the remaining rectangle has the same shape as the original one.



(a) When we considered the golden rectangle we had the equation

$$\frac{x}{1} = \frac{1}{x + 1} .$$

What is the corresponding equation for this rectangle?

(b) Start with $x = \frac{1}{2}$ and by successive trials find the value of x to at least two decimal places. Save your answer for later use.

22-4. Accuracy of the Approximations

In studying the golden rectangle, we have approximated x by rational numbers which are alternately too large and too small. The list of approximations begins as follows:

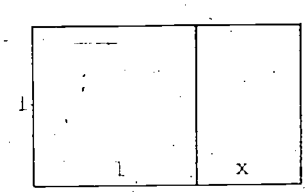
$$\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \ldots .$$

The required value x always lies between two successive numbers in this list. For example

(1) $\frac{3}{5} < x < \frac{5}{8}$.

Better yet

(2) $\frac{8}{13} < x < \frac{13}{21}$.



If we continue the list of rational approximations can we locate x as closely as we please? To answer this question, we try to find what happens to the difference between pairs of successive numbers in the list of rational approximations. We will see how far along the list we need to go to get an approximation of desired accuracy.

Since $\frac{5}{8} - \frac{3}{5} = \frac{5^2 - (3 \times 5)}{8 \times 5} = \frac{1}{40}$, we see that in (1) we have located x within an interval of length $\frac{1}{40}$.

$$\frac{13}{21} - \frac{8}{13} = \frac{13^2 - (8 \times 21)}{13 \times 21} = \frac{1}{273}$$

Thus in (2) we have located x within an interval of length $\frac{1}{273}$.

This is of course a much shorter interval. The question is: Can we make the interval as short as we please -- less than $\frac{1}{1,000,000}$, for example, or less than $\frac{1}{1,000,000,000}$?

To answer this question we must discover some general facts about the differences between two successive numbers in the list.

Did you notice that we were able to represent this difference by a fraction with numerator 1? Is this always true? Let us see where this numerator comes from. When we subtracted $\frac{3}{5}$ from $\frac{5}{8}$, the numerator was $5^2 - (3 \times 8)$. When we subtracted $\frac{8}{13}$ from $\frac{13}{21}$, the numerator was $13^2 - (8 \times 21)$.

In the Fibonacci series

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

5 is between 3 and 8, and 13 is between 8 and 21. We have already seen that the square of any Fibonacci number differs by 1 from the product of its neighbors.

Consequently we can write the list of differences between successive rational numbers in our list of approximations for x. This list of differences begins:

$$(3) \quad \frac{1}{1 \times 2}, \frac{1}{2 \times 3}, \frac{1}{3 \times 5}, \frac{1}{5 \times 8}, \frac{1}{8 \times 13}, \frac{1}{13 \times 21}, \ldots$$

Now let us look at the denominators in (3). We notice that they are the products of successive pairs of Fibonacci numbers

$$1 \times 2, \quad 2 \times 3, \quad 3 \times 5, \quad 5 \times 8, \ldots$$

These products increase as we progress along the list. In fact, we can show that each denominator is more than twice the preceding one. Proof: Two successive denominators are of the form $a \times b$ and $b \times c$. But $\frac{b \times c}{a \times b} = \frac{c}{a}$, and since $c > 2a$, we see that $\frac{c}{a} > 2$.

It follows that each difference is less than $\frac{1}{2}$ of the previous one. Therefore we can surely make the difference as small as we please by going out sufficiently far. For example, we know that

$$\frac{13}{21} - \frac{8}{13} = \frac{1}{273}.$$

The next difference is less than

$$\frac{1}{2 \times 273} \text{ or } \frac{1}{546},$$

the next difference is less than $\frac{1}{1000}$,

the next difference is less than $\frac{1}{2000}$

and so on.

Exercises 22-4

1. Given the list of Lucas numbers beginning

   1, 3, 4, 7, 11, 18, 29, 47,...

   we can write the list of fractions

   $\frac{1}{3}, \frac{3}{4}, \frac{4}{7}, \frac{7}{11}, \ldots$

   obtained by dividing each number by the following one. Consider the difference between two consecutive rational numbers in this list. Verify that when simplified, the numerator for each of the differences in the list is 5.

2. In Exercise 1 verify that each difference is less than one-half the preceding difference.

3. Consider the following list of rational numbers:

   $\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \frac{8}{21}, \ldots$

   (a) How are they related to the Fibonacci numbers

   1, 1, 2, 3, 5, 8, 13, 21, 34,...

(b) Write each number in the given list in decimal form and estimate to two decimal places the number (y) for which these rational numbers are successive approximations. (If necessary, continue the list.)

4. Find a relation between the list of rational numbers in Exercise 3, and the list

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \ldots$$

at the beginning of the section. Hint: Try multiplying adjacent pairs in the list given here.

5. Verify that your answer to Exercise 3 is approximately $1 - x$ where $x = .618\ldots$

22-5. The Solution in Terms of Square Roots

We have seen that the solution of

$$x = \frac{1}{x + 1}$$

is an irrational number. Since $\sqrt{2}$, $\sqrt{3}$, etc. are also irrational numbers it is natural to ask: Can x be expressed in terms of a square root? If

$$x = \frac{1}{x + 1}$$

it must be true that

$$x(x + 1) = 1.$$

We know that for all positive numbers a and b,

(1) $$\left(\frac{a + b}{2}\right)^2 = ab + \left(\frac{a - b}{2}\right)^2.$$

If we let $a = x + 1$ and $b = x$,

$$\frac{a + b}{2} = \frac{(x + 1) + x}{2} = x + \frac{1}{2}$$

and

$$\frac{a - b}{2} = \frac{(x + 1) - x}{2} = \frac{1}{2}.$$

Therefore (1) becomes

$$\left(x + \tfrac{1}{2}\right)^2 = x(x + 1) + \left(\tfrac{1}{2}\right)^2$$

$$= 1 + \tfrac{1}{4} = \tfrac{5}{4}$$

and therefore

$$x + \tfrac{1}{2} = \tfrac{\sqrt{5}}{2} \quad \text{or} \quad x + \tfrac{1}{2} = -\tfrac{\sqrt{5}}{2} .$$

Thus there are two solutions to the equation $x(x + 1) = 1$,

a positive solution, $-\frac{1}{2} + \frac{\sqrt{5}}{2}$,

and a negative solution, $-\frac{1}{2} - \frac{\sqrt{5}}{2}$.

Since $\sqrt{5} = 2.236\ldots$

we have

$$x = -.5 + 1.118\ldots \quad = .618\ldots$$

and

$$x = -.5 - 1.118\ldots \quad = -1.618\ldots$$

Of course we do not want the negative solution for the problem of the golden rectangle.

We can construct a segment which represents the positive solution $\frac{\sqrt{5}}{2} - \frac{1}{2}$ very simply as follows: Draw a right triangle ABC with base 1 and altitude $\frac{1}{2}$.

Its hypotenuse $\overline{AB}$ is

$$\sqrt{1 + \tfrac{1}{4}} = \sqrt{\tfrac{5}{4}} = \tfrac{\sqrt{5}}{2} .$$

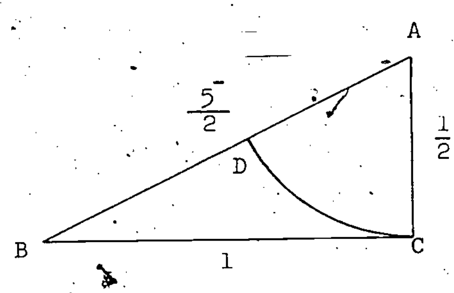


Figure 6

Draw a circle with center A and radius $AC = \frac{1}{2}$. Let D be the point of intersection of the circle and $\overline{AB}$.

Then $BD = AB - AD$

$$= \tfrac{\sqrt{5}}{2} - \tfrac{1}{2} .$$

Thus $\overline{BD}$ is the required segment.

There is another way in which we may discuss the solution of $x(x + 1) = 1$. We first change the equation to

$$x^2 + x = 1$$

and then to

$$x^2 = -x + 1.$$

We can solve this equation by finding the intersection of the graphs of

$$y = x^2$$

and

$$y = -x + 1.$$

As you know, the graph of $y = x^2$ is a parabola (see Figure 7) and the graph of

$$y = -x + 1$$

is a straight line with slope -1 and y-intercept 1.

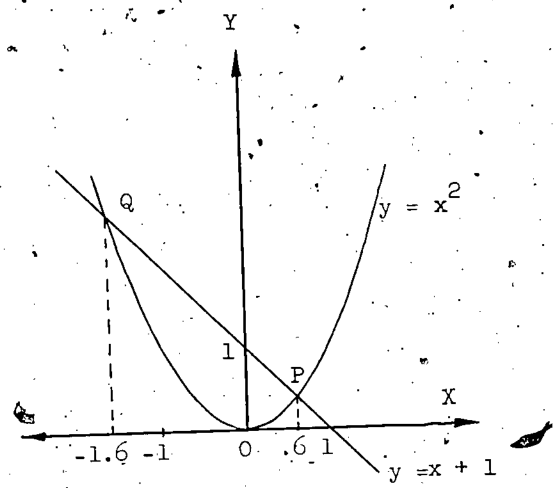


Figure 7

There is an intersection P near $x = .6$ and another intersection, Q near $x = -1.6$. Graphical solutions are necessarily approximate but they often give us a good start.

In this case, a little experimentation shows that $x = -1.6$ is about right. In fact

$$(-1.6)^2 = 2.56$$

and

$$-(-1.6) + 1 = 2.6.$$

Since the y-coordinate of the point on the line is greater than the y-coordinate of the point on the parabola, we know that -1.6 is to the right of the intersection or larger than the correct value.

If we try -1.62 we find $(-1.62)^2 = 2.6244$ and $-(-1.62) + 1 = 2.62$. Is -1.62 to the right or to the left of the intersection?

Exercises 22-5

1. Solve the following equations graphically:

   (a) $x(x + 1) = 2$ (Reminder: Change to the equation $x^2 = -x + 2$, and graph $y = x^2$ and $y = -x + 2$ on the same set of axes.)

   (b) $x(x + 1) = 3$

   (c) $x(x + 1) = 4$

   (d) $x(x + 1) = 6$

   (e) $x(x + 1) = -\frac{1}{4}$

2. Use the fact that $\left(\frac{a + b}{2}\right)^2 = ab + \left(\frac{a - b}{2}\right)^2$ to find the solutions of the equations in Exercise 1 above, indicating irrational solutions in terms of square roots. For example,

   If $x(x + 1) = 7$,

   we can let $a = x + 1$ and $b = x$, so $\frac{a + b}{2} = x + \frac{1}{2}$, and $\frac{a - b}{2} = \frac{1}{2}$.

   Since $ab = x(x + 1) = 7$, we have

   $$\left(x + \frac{1}{2}\right)^2 = 7 + \frac{1}{4} = \frac{29}{4}$$

   $$x + \frac{1}{2} = \frac{\sqrt{29}}{2} \quad \text{or} \quad x + \frac{1}{2} = -\frac{\sqrt{29}}{2}$$

   $$x = -\frac{1}{2} + \frac{\sqrt{29}}{2} \quad \text{or} \quad x = -\frac{1}{2} - \frac{\sqrt{29}}{2}.$$

   The solutions are: $-\frac{1}{2} + \frac{\sqrt{29}}{2}$, $-\frac{1}{2} - \frac{\sqrt{29}}{2}$.

3. Use the square root table of Chapter 21, Section 7, to find approximate solutions for the equations in Exercise 2, and check these with your results in Exercise 1.

Solve the following equations graphically. Also find the solutions in terms of square roots.

4. $x(x - 1) = 1$

5. $x^2 - x = 2$

6. $x^2 - x = 3$

7. $x^2 - x = 4$

8. In Exercises 22-3 (No. 3) we were led to solve the equation

$$x = \frac{1}{x + 2}$$

or $$x(x + 2) = 1.$$

Show that the solution can be expressed in terms of a square root. Use the square root table to check the accuracy of the answer previously obtained.

22-6. Projectiles

If a body is thrown straight up in the air with a speed of 64 ft/sec to begin with, we assume that the distance d feet above the ground t seconds later is given by the equation:

$$d = 64t - 16t^2.$$

This is a good model of the physical facts. Suppose that we ask the question: When is the body 48 ft. above the ground? To answer this question we set $d = 48$ and write

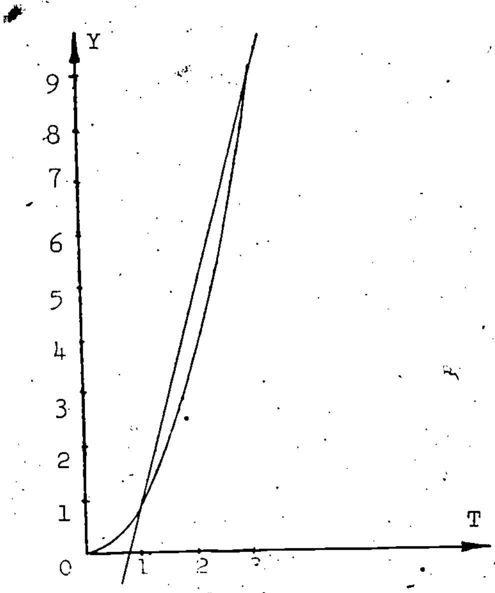
$$64t - 16t^2 = 48.$$

For what value or values of t is this true? If we divide by 16, we obtain the simpler equivalent equation

$$4t - t^2 = 3 \text{ or}$$

(1) $$t^2 = 4t - 3.$$

We can solve this graphically by finding the points of intersection of the parabola $y = t^2$ and the line $y = 4t - 3$.



The intersections occur at $(1, 1)$ and $(3, 9)$. It is easily verified that $t = 1$ and $t = 3$ actually satisfy equation (1). Why are there two answers?

If in the original question we had asked when the height was 16 feet, equation (1) would be replaced by

$$(2) \qquad t^2 = 4t - 1.$$

You will be asked to solve this equation graphically. Let us solve (2) in terms of square roots. We first rewrite (2) as

$$4t - t^2 = 1$$

and then as

$$t(4 - t) = 1.$$

Let $a = t$ and $b = 4 - t$ then $\left(\frac{a + b}{2}\right)^2 = ab + \left(\frac{a - b}{2}\right)^2$ gives

$$4 = 1 + (t - 2)^2.$$

So $\qquad (t - 2)^2 = 3.$

Then $\qquad t - 2 = \sqrt{3}$ or $t - 2 = -\sqrt{3}$

and $\qquad t = 2 + \sqrt{3}$ or $t = 2 - \sqrt{3}$.

Exercises 22-6

1. Find two approximate solutions of
   $t^2 = 4t - 1$ by drawing a graph of $y = t^2$ and $y = 4t - 1$.
2. Show that the results obtained in the text agree approximately with your graphical solutions in Exercise 1.
3. If a ball is thrown upward with a speed of 64 ft/sec (so that $d = 64t - 16t^2$) when does it reach the height 64 ft.? Solve graphically and also by the method of this section.
4. In Exercise 3, replace the height 64 ft., by 80 ft. Explain your failure to obtain an answer, either graphically or by the use of the formula.

22-7. Summary

In connection with the problem of finding the shape of the "golden rectangle" we are led to ask for a positive solution of $x = \frac{1}{1 + x}$. It is proved that it is impossible for x to be a rational number $\frac{p}{q}$. By considering the function

$$f : x \to \frac{1}{1 + x}$$

and starting with the input $x = 1$ and feeding back successive outputs as inputs we obtain a succession of approximations to the required solution. It is shown that these approximations are alternately too large and too small and that by continuing the process a sufficient number of times we can get as close as we please to the required solution.

It is also shown that the equation

$$x = \frac{1}{1 + x}$$

has the solutions $-\frac{1}{2} + \frac{\sqrt{5}}{2}$ and $-\frac{1}{2} - \frac{\sqrt{5}}{2}$, expressed in terms of square roots. The equation is also solved graphically by finding the intersections of a straight line with a parabola.

Chapter 23

SOLUTION SETS OF MATHEMATICAL SENTENCES

23-1. Solving Linear Equations

In the chapter on Problem Analysis, the following problem was stated and discussed:

> "In a gasoline economy test, one driver, starting with the first group of cars, drove for 5 hours at a certain speed and was then 120 miles from the finish line. Another driver, who set out later with a second group, had traveled at the same rate as the first driver for 3 hours and was then 250 miles from the finish. How fast were these two men driving?"

If we use r to represent the rate of a car, measured in miles per hour, we can represent certain functional relationships involved in the problem. The functions are listed below, with the output of each function described in words.

Function	Description of Output
$f : r \rightarrow 5r$	The distance that a car goes in five hours
$g : r \rightarrow 3r$	The distance that a car goes in three hours
$h : r \rightarrow 5r + 120$	120 miles more than the distance a car goes in five hours
$k : r \rightarrow 3r + 250$	250 miles more than the distance a car goes in three hours

We can express the fact that we are looking for a value of r for which the outputs of h and of k are equal by writing the mathematical sentence

$$5r + 120 = 3r + 250.$$

Each value of r for which the sentence is true is called a solution of the equation, and the set of all such values we call its solution set, or truth set.

One method of approach might be to draw the graphs of the two functions

$$f : r \to 5r + 120$$
$$g : r \to 3r + 250.$$

Since the slopes are not equal, we know that the graphs intersect. Thus we are sure that there is a solution, and only one solution.

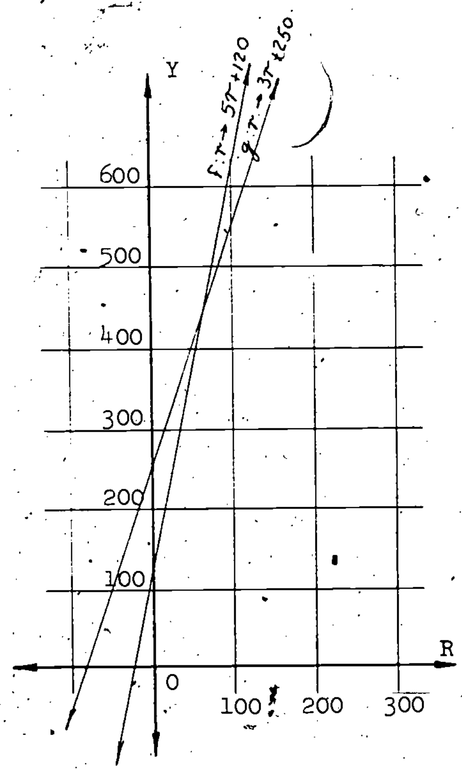
Exercises 23-1a

(Class Discussion)

1. Study the graphs of the functions

   $f : r \to 5r + 120,$
   $g : r \to 3r + 250,$

   shown here. Note that the rate is shown on the horizontal axis, and that the distance is shown on the vertical axis.



   (a) How many miles per hour is represented by the side of each square measured on the R-axis?

   (b) What do you estimate the value of r to be at the point of intersection of the two graphs?

   (c) Does your estimated value of r satisfy the equation

   $$5r + 120 = 3r + 250?$$

   (d) If it does not satisfy the equation, try some other numbers close to your estimate, until you <u>do</u> find the solution of the equation.

   (e) What does the solution of the equation mean in terms of the problem about the gasoline economy test?

In the exercises above, by graphing the two functions we found a solution for the equation

$$5r + 120 = 3r + 250$$

and were able to interpret the result to answer the question in the problem. However, the method was not very efficient, was it?

A more efficient way, often, is to write a chain, or list, of equations all with the same solution set as the given equation. If the last equation in the chain has an obvious solution set, then that is the solution set for the given equation. Let us consider how we can write such a chain.

For the two functions

$$f : r \to 5r + 120, \text{ and}$$
$$g : r \to 3r + 250,$$

the domain of each function is the set of all real numbers. That is, if any real number is used as input for the function, the output will be a single real number. With respect to the equation, we say that the set of all real numbers is the replacement set for the equation $5r + 120 = 3r + 25$.

The use of any property that holds for all members of the replacement set of an equation enables us to write another equation whose solution set is the same as for the first equation. The field properties (commutative, associative, distributive, etc.) are true for all real numbers; the replacement set of any equation that we shall be using will be some subset of the set of real numbers. Hence the use of any of the field properties will give us another equation with the same solution set.

Equations which have the same replacement set and the same solution set are called equivalent equations. A convenient symbol for "is equivalent to" is " $\Longleftrightarrow$." For example, we write

$$5x + 2x = 35 \Longleftrightarrow (5 + 2)x = 35$$

to mean that if there is a real number x for which "$5x + 2x = 35$" is a true statement, then "$(5 + 2)x = 35$" is true for the same value of x, and vice versa.

In addition to the field properties, there are two more "properties" which are consequences of our agreement that "$a = b$" means that "a" and "b" name the same number. We reason as follows:

If a and c are numbers, then "$a + c$" is a name for the sum of these two numbers, and "ac" is a name for their product.

If the first number is called "b" instead of "a", then "$b + c$" is a new name for the sum, and "bc" is a new name for the product.

Thus "$a + c$" and "$b + c$" both name the same sum, and we write "$a + c = b + c$"; similarly, both "ac" and "bc" name the same product, and we write "$ac = bc$."

Check Your Reading

1. If you solve an equation by writing a chain of equivalent equations, what should be true about the solution set of the last equation in the chain?

2. What is the replacement set for the equation $5r + 120 = 3r + 25$? What is the solution set?

3. If two equations are equivalent, what two things are true about them? What symbol is used for "is equivalent to"?

4. Why can the field properties be used to write equations equivalent to a given equation?

5. If a, b, and c are real numbers such that $a = b$, what is true about $a + c$ and $b + c$? about ac and bc?

Exercises 23-1b

(Class Discussion)

1. (a) We have shown that, for a, b, and c real numbers, if $a = b$ then $a + c = b + c$. How could you show that if $a + c = b + c$, then $a = b$?

   (b) Write an "if and only if" statement which combines both statements.

2. (a) We also saw that, for a, b, and c real numbers, if $a = b$ then $ac = bc$. What restriction on c is necessary for "if $ac = bc$, then $a = b$" to be a true statement?

   (b) Write an "if and only if" statement for multiplication and equality.

Thus we have two "properties" of equality which hold for real numbers, and so can be used to write equivalent equations:

(1) For all real numbers a, b, and c, $a = b$ if and only if $a + c = b + c$.

(2) For all real numbers a, b, and c such that $c \neq 0$, $a = b$ if and only if $ac = bc$.

For convenience, we shall refer to (1) as the <u>addition property of equality</u> and to (2) as the <u>multiplication property of equality</u>.

For the equation $5r + 120 = 3r + 250$, then, we could have written a chain of equivalent equations as follows:

$$
\begin{array}{llr}
(1)\quad 5r + 120 = 3r + 250 & \Longleftrightarrow \; 5r + 120 + (-120) = 3r + 250 + (-120) & (2) \\
& \Longleftrightarrow \; 5r = 3r + 130 & (3) \\
& \Longleftrightarrow \; 5r + (-3r) = 130 & (4) \\
& \Longleftrightarrow \; 2r = 130 & (5) \\
& \Longleftrightarrow \; r = 65 & (6)
\end{array}
$$

The solution set of equation (6) is obviously $\{65\}$, and we have used properties which hold for all real numbers. Hence we know that the solution set of equation (1) is also $\{65\}$.

Note that the symbol "$\Longleftrightarrow$" and its meaning "is equivalent to" are other ways of saying "if and only if." For example,

$$x + 3 = 5 \Longleftrightarrow x = 2$$

could be stated: "$x + 3 = 5$ is true <u>if and only if</u> $x = 2$ is true." That is, if $x + 3 = 5$, then $x = 2$, and if $x = 2$, then $x + 3 = 5$.

<u>Exercises 23-1c</u>

1. For each equation in the chain of equations equivalent to $5r + 120 = 3r + 250$, tell which property applies.

2. Complete each of the following so that the two equations are equivalent, and indicate which property of equality has been used.

   (a) $x = 7 \Longleftrightarrow x + 3 =$ ____

   (b) $5x = 12 \Longleftrightarrow x =$ ____

   (c) $x - .02 = 3 \Longleftrightarrow x =$ ____

   (d) $10.3 = x + 4 \Longleftrightarrow$ ____ $= x$

   (e) $27 = \frac{3}{5}x \Longleftrightarrow$ ____ $= x$

(f) $C = 2\pi r \Longleftrightarrow r =$ ______

(g) $7x = 6x - 3 \Longleftrightarrow x =$ ______

(h) $p = a + b + c \Longleftrightarrow$ ______ $= b$

3. State the solution set of each of the following equations. (Show the steps you take, and indicate where you use the addition and multiplication properties of equality.)

(a) $3x + 7 = 16$

(b) $\frac{1}{2}x - 2 = 2$

(c) $8x - 4 = 5x + 9$

(d) $.03x + 12 = .05x + 3$

(e) $.03(x + 12) = .05(x + 3)$

(f) $7x + 10 - x = 3(x + 1)$

4. The addition and multiplication properties of equality can be used in changing a formula from one form to another. For example, the equation $F = \frac{9}{5}C + 32$ is the conversion formula used in changing a temperature measurement from Centigrade degrees to Fahrenheit degrees. The same formula could be used to change Fahrenheit degrees to Centigrade degrees. However, it is more convenient to use a formula which is obtained as follows:

If $F = \frac{9}{5}C + 32$,

then $F - 32 = \frac{9}{5}C$, by the addition property of equality

or $\frac{9}{5}C = F - 32$,

and $C = \frac{5}{9}(F - 32)$, by the multiplication property of equality.

(a) Name the following temperature measurements in Fahrenheit degrees: $0^\circ C$; $60^\circ C$; $100^\circ C$; $-40^\circ C$.

(b) Name the following temperature measurements in Centigrade degrees: $0^\circ F$; $23^\circ F$; $59^\circ F$; $104^\circ F$.

5. Use the addition and multiplication properties of equality to write formulas as indicated; in (a)-(c) use whichever formula is more convenient to find the values indicated.

(a) If $P = 2\ell + 2w$, then $w =$ _____.

If $\ell = 7\frac{1}{2}$ in. and $w = 3\frac{3}{4}$ in., then $P =$ _____.

If $P = 18$ ft. and $\ell = 5.7$ ft., then $w =$ _____.

(b) If $L = 2\pi r h$, then $r =$ _____.

Find L if $r = 3.5$ in. and $h = 7$ in. (use $\pi \approx 3\frac{1}{7}$).

Find r if $L = 330$ sq. ft. and $h = 42$ ft. (use $\pi \approx 3\frac{1}{7}$).

(c) If $A = p + prt$, then $r =$ _____.

Find A if $p = 200$ (dollars), $r = 3\%$, and $t = 5$ (years).

If $A = 168$ (dollars), $p = 150$ (dollars), and $t = 3$ (years), then $r =$ _____%.

(d) If $s = at - \frac{1}{2}gt^2$, then $a =$ _____.

(e) If $E = I(R + r)$, then $R =$ _____.

As we have seen, an equation can serve as a model for the situation in a problem. The solution of the equation indicates the answer to the question raised in the problem.

Example

"The distance an object falls during the first second is 32 feet less than the distance it falls during the second second. During the two seconds it falls 48 feet. How far does it fall during the second second?"

If d represents a number of feet that an object falls during the second second, we can write the following functional relationships involved in the problem:

Function	Description of Output
$f : d \to d - 32$	The distance an object falls during the first second
$g : d \to (d - 32) + d$	The distance an object falls during two seconds

We are looking for a value of $\underline{d}$ such that the output of g is 48. Thus an equation which serves as a model for the situation is

$$(d - 32) + d = 48.$$

$$\begin{aligned}(d - 32) + d = 48 &\iff d + d - 32 = 48\\ &\iff 2d - 32 = 48\\ &\iff 2d = 80\\ &\iff d = 40.\end{aligned}$$

The solution set of each equation is $\{40\}$. Hence the object falls 40 feet during the second second.

Exercises 23-1d

For each of the following problems, (a) analyze the situation and write an equation which is a suitable model, (b) solve the equation, and (c) interpret the solution and answer the question in the problem.

1. If you take one-third of a number, you get the same result as if you subtract 9 from one-half the same number. What is the number?

2. The degree measure of the largest angle of a triangle is 15 more than twice the degree measure of the smallest angle. The degree measure of the third angle is 10 less than twice the degree measure of the smallest angle. What is the measure of the smallest angle?

3. John has 15 feet of fencing. He plans to use it to enclose a rectangular garden 3 ft. wide.
   (1) How long can he make the garden, if he uses all of the fencing?
   (2) Why will the shape of the garden not be a "golden rectangle"?
   (3) If he had one more foot of fencing would the shape then be a "golden rectangle"?

4. The amount of $205 is to be divided among Tom, Dick, and Harry. Dick is to have $15 more than Harry, and Tom is to have twice as much as Dick. How must the money be divided?

5. A square and an equilateral triangle have equal perimeters. A side of the triangle is five inches longer than a side of the square. What is the length of the side of the square?

6. Mr. Barton paid $\$176$ for a freezer. The price he paid was at a discount of 12% of the marked price. What was the marked price?

7. John Jones's total pay was $\$166.40$ for a week in which he worked 48 hours. He is paid "overtime" for all hours over 40 hours, at the rate of $\frac{3}{2}$ times his normal rate. What was his normal rate of pay per hour?

23-2. Solving Linear Inequalities

Consider this problem: If John had $\$2$ more than twice the amount he now has, he would still have less than $\$10$. What do you know about the amount of money he now has?

If we use x to represent the number of dollars John might have, then the following functional relationships are involved in the problem.

Function	Description of Output
$x \to 2x$	Twice the number of dollars John might have
$x \to 2x + 2$	2 more than twice the number of dollars he might have

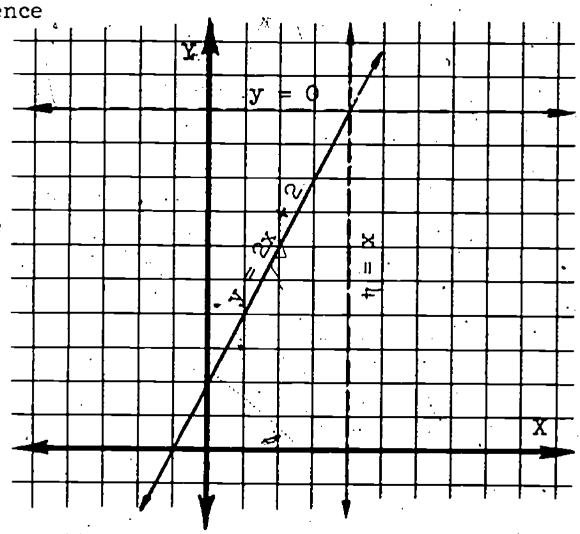
The phrase "less than $\$10$" suggests the relationship represented by the inequality

$$2x + 2 < 10.$$

The problem implies that John has <u>some</u> money, a fact that can be represented by the inequality "$x > 0$." Thus the situation in the problem can be modeled by the compound sentence

$2x + 2 < 10$ and $x > 0$.

One way of determining the solution set of this sentence, and thus answering the question in the problem, is by means of a graph like the one shown here.



Exercises 23-2a

(Class Discussion)

1. The point $(4, 10)$ is the intersection of the graphs of "$y = 2x + 2$" and of "$y = 10$." At this point, then, $2x + 2 = 10$, and hence $x = 4$.

   (a) Between what two points on the graph of $y = 2x + 2$ is it true that $y < 10$ and $x > 0$?

   (b) Write a compound open sentence which describes the abscissas of all of the points on the line between the two points you described in (a).

   (c) Use set-builder notation to state the solution set of the compound sentence

   $$2x + 2 < 10 \text{ and } x > 0.$$

   (d) Use the solution set to help you answer the question in the problem.

2. Suppose that the problem had specified that his number of dollars was an integer.

   (a) What would be the graph in the coordinate plane of the compound sentence with this added restriction?

   (b) What is the solution set of the sentence

   $$2x + 2 < 10 \text{ and } x > 0 \text{ and } x \text{ is an integer?}$$

   (c) With this restriction, answer the question "What do you know about the amount of money John now has?"

For solving many equations, a more efficient method than the use of a graph is the writing of a chain of equivalent equations, with the final equation having a solution set that is obvious. Is there a similar method for inequalities?

We define <u>equivalent inequalities</u> as inequalities which have the same replacement set and the same solution set, and we continue to use the same symbol, "$\Longleftrightarrow$", for "is equivalent to." In the case of equations, we wrote equivalent equations by using the field properties, and the addition and multiplication properties of equality.

You may recall that you have already used the Addition Property of Order and the Multiplication Property of Order for real numbers. These, together with the field properties, enable us to write chains of inequalities. In terms of the order relation "$<$", they can be stated:

<u>Addition Property of Order</u>: For all real numbers a, b, and c,
$a < b$ if and only if $a + c < b + c$.

<u>Multiplication Property of Order</u>: For all real numbers a, b, and c such that $c \neq 0$,

(1) $a < b$ if and only if $0 < c$ and $ac < bc$.

(2) $a < b$ if and only if $c < 0$ and $bc < ac$.

Since "$b > a$" can be equivalently expressed as "$a < b$", similar statements can be made in terms of the relation "$>$".

For the inequality $2x + 2 < 10$, we could find the solution set by writing the following chain of equivalent inequalities:

$$2x + 2 < 10 \iff 2x < 8$$
$$\iff x < 4.$$

$\{x : x < 4\}$ is the solution set of the final inequality so it is also the solution set of $2x + 2 < 10$. For the compound sentence

$$2x + 2 < 10 \text{ and } x > 0$$

we have as solution set

$$\{x : 0 < x < 4\}.$$

<u>Exercises 23-2b</u>

Use a chain of equivalent inequalities to find the solution set of each inequality.

Example: $2x + 5 \leq 11 \iff 2x \leq 6$
$\iff x \leq 3$

The solution set is $\{x : x \leq 3\}$.

1. (a) $x + 7 < 10$
   (b) $x - 2 < 5$
   (c) $x + .9 < 3.2$
   (d) $x + .03 > .03$
   (e) $3x + 5 > 2x + 4$
   (f) $x - \frac{9}{2} \leq \frac{5}{3}$
   (g) $x + 5\frac{1}{4} \geq 9\frac{1}{2}$
   (h) $5x - 2 \leq 4x + .04$

2. (a) $3x < 5$

(b) $\frac{1}{3}x \leq 5$

(c) $-3x < 5$

(d) $.4x > 6$

(e) $1.2x \geq .6$

(f) $-1.5x > 0.75$

(g) $-\frac{1}{5}x \geq -2$

(h) $17x < -23$

3. (a) $2x + 3 > 5$

(b) $2x + 3 > -5$

(c) $-2x + 3 < -5$

(d) $-2x + 1 < 5$

(e) $5 - 2x < 4x - 3$

(f) $-(2 + x) > 3 - 7$

(g) $-2 + 5 - 3x > 4x + 7 - 2x$

4. (a) Draw number line graphs for the following:

$$|x| = 5$$
$$|x| < 5$$
$$|x| > 5$$

(b) Draw a number line graph and state the solution set for each of the following:

$$|x| + 2 \leq 5$$
$$|x| + 2 \geq 5$$

5. For each of the following, find the values of x for which the statement is true.

(a) $5 - 3 < x$

(b) $3 - 5 < x$

(c) $|5 - 3| < x$

(d) $|3 - 5| < x$

For each of the following problems, (a) analyze the situation and state an inequality which is a suitable model; (b) solve the inequality; and (c) interpret the solution and answer the question in the problem.

6. The body of a certain missile is eleven times the length of its nose cone. The total length of the missile is at least 100 feet. How long must the nose cone be? (Note that you cannot answer this with a single number.)

7. Two cars start from the same point, at the same time, and travel in opposite directions. One car travels 10 miles per hour faster than the other. At the end of 3 hours they are more than 200 miles apart. What do you know about the rate of each car?

8. If a number of flower bulbs of a certain type are planted, it is known that fewer than $\frac{7}{8}$ of them will grow. However, with proper care, at least $\frac{1}{2}$ of them will do well. If a careful gardener grows 18 of these bulbs, how many did he probably plant?

9. Jim receives \$1.75 per hour for work which he does in his spare time. He is saving his money to buy a car which will cost him at least \$75. What is the smallest integral number of hours he must work?

The Multiplication Property of Order can be used to help us discover and prove other interesting and useful facts about order. For example, if you know the order of two numbers, what do you know about the order of their reciprocals?

Exercises 23-2c

(Class Discussion)

1. $5 < 8$ is true and $\frac{1}{8} < \frac{1}{5}$ is true.

   $2 < 3$ is true and $\frac{1}{3} < \frac{1}{2}$ is true.

   (a) Do you think then, that for all real numbers a and b, if $a < b$, then $\frac{1}{b} < \frac{1}{a}$? What about -3 and 2?

   $-3 < 2$ is true; is $\frac{1}{2} < -\frac{1}{3}$ also true?

   What about -3 and -2?

   $-3 < -2$ is true; is $-\frac{1}{2} < -\frac{1}{3}$ also true?

   (b) Thus we must consider three cases:

   (1) If $a < b$ and both are positive,

   (2) If $a < b$ and both are ____,

   (3) If $a < b$, with a negative and b ____.

   Why do we not need to consider the case of $a < b$, with a positive and b negative? Why must a and b be nonzero?

(c) Complete these proofs:

(1) Suppose that a and b are positive real numbers such that $a < b$.

$a > 0$, hence $\frac{1}{a} \underset{(>,<,=)}{\underline{\qquad}} 0$; $b > 0$, hence $\frac{1}{b} \underset{(>,<,=)}{\underline{\qquad}} 0$.

Thus, $\frac{1}{a} \cdot \frac{1}{b} \underset{(>,<,=)}{\underline{\qquad}} 0$.

Since $a < b$ and $\frac{1}{a} \cdot \frac{1}{b} > 0$,

$a(\frac{1}{a} \cdot \frac{1}{b}) < b(\underline{\qquad\qquad})$

$\underline{\qquad\qquad} < \underline{\qquad\qquad}$.

(2) Suppose that a and b are negative real numbers such that $a < b$.

$a < 0$, hence $\frac{1}{a} \underline{\qquad\qquad}$

$b < 0$, hence $\frac{1}{b} \underline{\qquad\qquad}$

$\frac{1}{a} \cdot \frac{1}{b} \underline{\qquad\qquad} 0$.

(You complete the proof.)

(3) Suppose that a is a negative real number, and b is a positive real number. Then $a < 0$, and $0 < b$, hence

$a \underline{\qquad} b$.

If $a < 0$, then $\frac{1}{a} \underline{\qquad} 0$; if $0 < b$, then $0 \underline{\qquad} \frac{1}{b}$

$\frac{1}{a} \underline{\qquad} \frac{1}{b}$.

What we have just proved can be stated as a theorem about the order of the reciprocals of two numbers:

For any two nonzero real numbers a and b, if $a < b$, then

$\frac{1}{b} < \frac{1}{a}$ if both a and b are positive or both a and b are negative,

$\frac{1}{a} < \frac{1}{b}$, if a is negative and b is positive.

Exercises 23-2d

1. State and prove a general property about the order of the opposites of two numbers a and b, such that $a < b$.

2. If $a < b$, with a and b both positive, prove that $a^2 < b^2$.

3. If $a < b$, with a and b both negative, what can you prove about the order of a^2 and b^2?

4. If $a < b$, with a negative and b positive, what can you prove about the order of a^2 and b^2? If you also know that $|a| < |b|$, what can you prove about the order of a^2 and b^2?

5. If $a \neq 0$, prove that $a^2 > 0$.

23-3. Solving Fractional Equations

Suppose that we consider finding a solution to the following problem, by writing an equation to serve as a model, and then solving the equation.

> One pump fills a certain tank twice as fast as a smaller pump does. If they work together they fill the tank in 16 minutes. How long does the larger pump require if it works alone?

If x represents a number of minutes a pump requires to fill a certain tank, we can write these functional relationships:

Function	Description of Output
$f : x \rightarrow \frac{1}{2} x$	The number of minutes required by a pump which fills a certain tank twice as fast as a pump which requires x minutes
$g : x \rightarrow \frac{1}{x}\ (x \neq 0)$	The part of the tank filled in one minute by a pump which requires x minutes to fill the entire tank
$h : x \rightarrow \frac{1}{\frac{1}{2}x}\ (x \neq 0)$	The part of the tank filled in one minute by a pump which requires $\frac{1}{2}x$ minutes to fill the entire tank

$F : x \rightarrow \frac{1}{x} + \frac{1}{\frac{1}{2}x}$ $(x \neq 0)$ The part of the tank filled in one minute by both pumps working together

If it takes the two pumps working together 16 minutes to fill the tank, we can think of the result as the same as for one "super-pump" working for 16 minutes. The output of function

$$g : x \rightarrow \frac{1}{x} \quad (x \neq 0)$$

for an input of 16 is $\frac{1}{16}$. Hence the "super-pump," or the combination of the two pumps, fills $\frac{1}{16}$ of the tank in 1 minute.

Thus, in order to answer the question in the problem, we need to find a value of x for which the output of the function

$$F : x \rightarrow \frac{1}{x} + \frac{1}{\frac{1}{2}x} \qquad (x \neq 0)$$

is $\frac{1}{16}$. We can state this as the compound sentence

$$\frac{1}{x} + \frac{1}{\frac{1}{2}x} = \frac{1}{16} \text{ and } x \neq 0.$$

Exercises 23-3a

(Class Discussion)

1. What is the domain of each of functions g, h, and F? Why? What, then, is the replacement set for the equation

$$\frac{1}{x} + \frac{1}{\frac{1}{2}x} = \frac{1}{16} \text{ and } x \neq 0?$$

2. Complete this chain of equivalent equations.

$$\frac{1}{x} + \frac{1}{\frac{1}{2}x} = \frac{1}{16} \text{ and } x \neq 0$$

$$\Longleftrightarrow \frac{1}{x} + \frac{(\ )}{x} = \frac{1}{16} \text{ and } x \neq 0$$

$$\Longleftrightarrow \frac{(\ )}{x} = \frac{1}{16} \text{ and } x = 0$$

$$\Longleftrightarrow x = \_\_\_\_\_ \text{ and } \_\_\_\_\_.$$

3. State the solution set of the sentence

$$\frac{1}{x} + \frac{1}{\frac{1}{2}x} = \frac{1}{16} \text{ and } x \neq 0.$$

4. How long does the larger pump require to fill the tank?

____

In the discussion above, note that all of the compound sentences have the same replacement set, $\{x : x \neq 0\}$. That is, the replacement set for the sentence is the domain of the function

$$F : x \to \frac{1}{x} + \frac{1}{\frac{1}{2}x} \quad (x \neq 0).$$

The value 0 for x is excluded from the domain of the function. This is because the number 0 has no reciprocal, and hence $\frac{1}{x}$ has no meaning for $x = 0$. In fact, the domain of a function cannot include any values of the variable for which the function is not defined.

Exercises 23-3b

1. For each of the following expressions, indicate the values of the variable for which the expression could not define a function.

(a) $\frac{1}{x + 7}$

(b) $\frac{1}{x^2 + 7}$

(c) $\frac{1}{x^2 - 7}$

(d) $\frac{1}{|x| + 7}$

(e) $\frac{1}{\sqrt{x} + 7}$

(f) $\frac{1}{(x - 1)(x + 3)}$

(g) $\frac{x}{x(x + 5)}$

(h) $\frac{x}{x^2 + 5}$

(i) $\frac{x}{|x|}$

(j) $\frac{x^2}{|x| - 3}$

(k) $\frac{x^2}{|x| + 3}$

2. For each of the following, the replacement set is not the set of all real numbers. Replace each by a compound sentence which indicates the replacement set.

Example: $\frac{1}{x + 2} = 6$ has no meaning for $x = -2$, so we write the compound sentence

$$\frac{1}{x + 2} = 6 \text{ and } x \neq -2.$$

(a) $\frac{5}{x + 1} = 3$

(b) $\frac{y + 1}{|y| - 2} = 3$

(c) $2\sqrt{x} = \frac{x + 1}{\sqrt{x}}$

(d) $\frac{2}{3y} - \frac{1}{3} = \frac{2 + 3y}{y}$

(e) $\frac{3}{t} = \frac{1}{t + 1}$

With the restrictions on the replacement set stated by writing a compound sentence, we can write a chain of equivalent sentences for any fractional equation, that is, for any equation involving a fraction containing a variable in its denominator. To do this, we again use the field properties and the addition and multiplication properties of equality.

Example 1. Solve $\frac{5}{x - 1} = 3$ and $x \neq 1$.

$$\begin{aligned} \frac{5}{x - 1} = 3 \text{ and } x \neq 1 &\iff (x - 1) \cdot \frac{5}{x - 1} = 3(x - 1) \text{ and } x \neq 1 \\ &\iff 5 = 3x - 3 \text{ and } x \neq 1 \\ &\iff 3x = 8 \text{ and } x \neq 1 \\ &\iff x = \frac{8}{3} \text{ and } x \neq 1 \end{aligned}$$

The solution set is $\{\frac{8}{3}\}$.

Example 2. Solve $\frac{2}{n} = \frac{2}{n + 1}$ and $n \neq 0$, $n \neq -1$.

$$\begin{aligned} \frac{2}{n} = \frac{2}{n + 1} \text{ and } n \neq 0, n \neq -1 &\iff 2(n + 1) = 2n \text{ and } n \neq 0, n \neq -1 \\ &\iff 2n + 2 = 2n \text{ and } n \neq 0, n \neq -1 \\ &\iff 2 = 0 \text{ and } n \neq 0, n \neq -1 \end{aligned}$$

Since there is no value of n for which $2 = 0$ is a true statement, the solution set is $\emptyset$.

Example 3. Solve $\frac{x}{x-3} = \frac{3}{x-3}$ and $x \neq 3$.

$$\frac{x}{x-3} = \frac{3}{x-3} \text{ and } x \neq 3 \iff (x-3)\frac{x}{x-3} = (x-3)\frac{3}{x-3} \text{ and } x \neq 3$$

$$\iff x = 3 \text{ and } x \neq 3$$

Since there is no value of x for which both parts of the compound sentence $x = 3$ and $x \neq 3$ are true, the solution set is $\emptyset$.

Exercises 23-3c

1. By writing a chain of equivalences, determine the solution set of each of the following:

(a) $\frac{2}{x} - \frac{3}{x} = 10$ and $x \neq 0$

(b) $\frac{x}{2} - \frac{x}{3} = 10$

(c) $x + \frac{1}{x} = \frac{2}{x}$ and $x \neq 0$

(d) $x + \frac{2}{x} = \frac{1}{x}$ and $x \neq 0$

(e) $\frac{1}{t} = \frac{1}{t-1}$ and $t \neq 0, t \neq 1$

(f) $\frac{1}{t} = \frac{1}{1-t}$ and $t \neq 0, t \neq 1$

(g) $\frac{2}{t} = \frac{1}{t-1}$ and $t \neq 0, t \neq 1$

2. For each of the following, if the replacement set is not the set of all real numbers, write a compound sentence by adding a clause or clauses restricting the replacement set. Then use a chain of equivalent sentences to determine the solution set:

(a) $\frac{5}{x+1} = 4$

(b) $\frac{5}{x^2+1} = 4$

(c) $\frac{x^2+5}{x^2+5} = 0$

(d) $\frac{x^2+5}{x^2+5} = 1$

(e) $\frac{4}{3} - \frac{y}{5} = \frac{1}{2}$

(f) $\frac{y}{y-2} = 3$

(g) $\frac{2x^2+5}{x^2-5} = 1$

(h) $2\sqrt{x} = \frac{x+1}{\sqrt{x}}$

(i) $\frac{5}{2}x - 17 = 33$

(j) $\frac{15}{|x|+2} = 3$

(k) $\frac{5}{|x|+2} = 3$

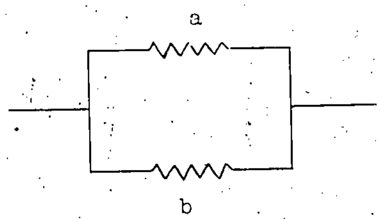
(ℓ) $4x + \frac{3}{2} = x + 6$

(m) $\frac{1}{x} + 3 = \frac{2}{x}$

For a variety of problems applying fractional equations, the equation takes the general form $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. An instance of this which you have already seen is the problem with which this section started, about two pumps filling a tank. To solve the problem, we wrote the compound sentence

$\frac{1}{x} + \frac{1}{\frac{1}{2}x} = \frac{1}{16}$ and $x \neq 0$, and found its solution set.

Another application is in electricity. If two resistances, of $\underline{a}$ ohms and $\underline{b}$ ohms, respectively are connected in parallel, the total resistance, c ohms, of the circuit is given by the equation



$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Exercises 23-3d
(Class Discussion)

In an electrical circuit in which two resistances are connected in parallel, the larger resistance is three times the smaller resistance. The total resistance is 3 ohms. What is the smaller resistance?

1. If r represents a number of ohms resistance, complete these statements of functional relationships suggested by the problem:

Function	Description of Output
$f : r \rightarrow$ ____	A number of ohms resistance which is three times a resistance of r ohms
$g : r \rightarrow$ ____	The reciprocal of r
$h : r \rightarrow$ ____	The reciprocal of $3r$
$F : r \rightarrow \frac{1}{r} + \frac{1}{3r}$	The sum of the reciprocals of r and $3r$

2. (a) The total resistance for the circuit is 3 ohms. Write an equation which states that the output of $F : r \rightarrow \frac{1}{r} + \frac{1}{3r}$ is the reciprocal of 3.
   (b) What is the replacement set for the equation?
   (c) Write a compound sentence combining the information in parts (a) and (b).

3. (a) Use a chain of equivalent sentences to determine the solution set of the compound sentence in 2(c).

(b) How many ohms are in the smaller resistance?

The next set of exercises includes some problems for which a useful mathematical model is an equation of the general form

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

For other problems, you may find a different form of equation. In each case, looking for functional relationships will help you find a suitable equation to use.

Exercises 23-3c

For each of the following problems (a) write a mathematical sentence which is a suitable model for the situation, (b) find the solution set of the sentence, and (c) answer the question in the problem.

1. Printing press A can do a certain job in 3 hours, and press B can do the same job in 2 hours. If both presses work on the job at the same time, in how many hours will they complete it?

2. One bulldozer can clear land twice as fast as a smaller one. Together they clear a large tract in $1\frac{1}{2}$ hours. How long would the larger bulldozer alone take?

3. Air conditioner A is found to lower the temperature of a room 10 degrees in the first 12 minutes. With air conditioner B working with A, the first change of 10° takes [illegible] minutes. How long would the device B alone need to produce an initial change of 10°?

4. In a certain school, the ratio of boys to girls was $\frac{7}{6}$. If there were 2600 students in the school, how many girls were there?

5. A certain mixture for killing weeds must be made in the ratio of 3 parts of weed-killer to 17 parts of water. How many quarts of weed-killer should be put in a 10 gallon tank which is going to be filled up with water to make 10 gallons of mixture?

6. Don averaged 36 m.p.h. when driving to work, and 30 m.p.h. when driving home on the same route. His time returning was 18 minutes more than his time going to work. Find the distance, along the route,

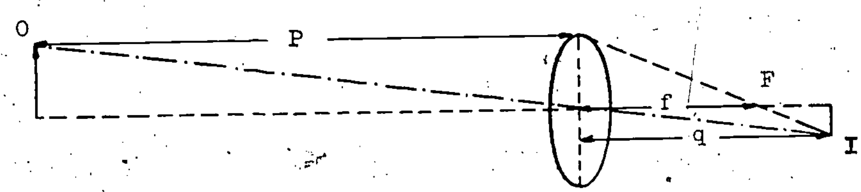
from his home to his work.

Suggestion: Use the fact that time = $\frac{\text{distance}}{\text{rate}}$.

7. A troop of scouts hiked a distance of 15 miles to the council scout cabin. They returned in cars over the same road at an average rate of 30 miles an hour. If the round trip had to be made in not more than $5\frac{1}{2}$ hours, at what rate did they have to hike out?

8. Two resistances are connected in parallel in an electrical circuit. The smaller of the resistances is 2 ohms less than the larger, and the total resistance equals $\frac{5}{8}$ of the smaller resistance. Find the smaller resistance.

9. Another type of problem for which we have a mathematical model of the form $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ is one which involves a simple lens, as pictured here.



If f represents the focal length of a lens, p represents the distance of an object from the lens, and q represents the distance of the image of the same object from the lens, then

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}.$$

If the focal length of the lens of a camera is $\frac{1}{2}$ in., at what distance from the lens will the image of an object lie if the distance of the object from the lens is 10 feet?

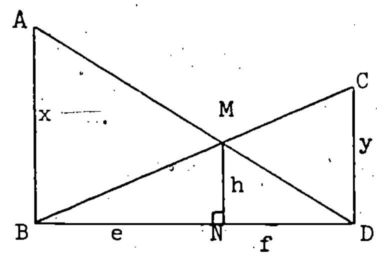
10. (1) If you were building a camera such that the distance from the lens to the film would be 0.6 in., what focal length should the lens have so that you will get a distinct image of an object which is at a distance of 10 feet from the camera?

(2) What focal length would you need to get a distinct image at 20 feet? at 5 feet?

(3) Express the focal lengths that you found in parts (1) and (2) as 3-place decimals, and comment.

11. In the figure shown here, with $\overline{AB}$, $\overline{CD}$, and $\overline{MN}$ all $\perp$ to $\overline{BD}$, we can prove that

$$\frac{1}{h} = \frac{1}{x} + \frac{1}{y}.$$



(a) Complete the proof:

$\triangle DMN \sim \triangle DAB$, hence $\frac{h}{(\quad)} = \frac{f}{e + f}$

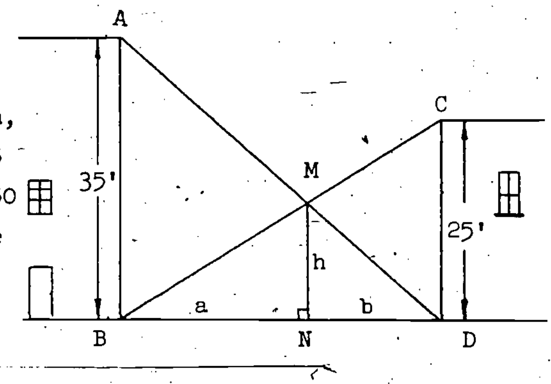
$\triangle BMN \sim \triangle$___, hence $\frac{(\quad)}{y} = \frac{e}{(\quad)}$

$\frac{h}{x} + \frac{h}{y} = \frac{(\quad)}{e + f}$

$\frac{1}{x} + \frac{1}{y} =$ ____

(b) Explain why the length of $\overline{BD}$ has no part in the relationship.

12. Suppose that $\overline{AB}$ and $\overline{CD}$ are corners of two houses, and that wires are stretched from A to D and from B to C, as shown, intersecting at M. If AB = 35 feet, CD = 25 feet and BD = 50 feet, find h, the height above the ground of point M.



23-4. Inequalities Involving Fractions

We defined equivalent inequalities as inequalities which have the same replacement set and the same solution set. In writing chains of equivalent inequalities we can use the field properties as well as the Addition and Multiplication Properties of Order.

However, if we have an inequality which involves a fraction whose denominator contains the variable, special care is needed in applying the Multiplication Property of Order. Separate results must be considered for the case in which the multiplier is positive and that in which it is negative.

Example: Solve $\frac{2}{x} < \frac{1}{x} + 3$, and $x \neq 0$.

Case 1: $x > 0$

$$\frac{2}{x} < \frac{1}{x} + 3 \text{ and } x > 0 \iff \frac{1}{x} < 3 \text{ and } x > 0$$
$$\iff 1 < 3x \text{ and } x > 0$$
$$\iff \frac{1}{3} < x \text{ and } x > 0$$

Solution set: $\{x : x > \frac{1}{3}\}$

Case 2: $x < 0$

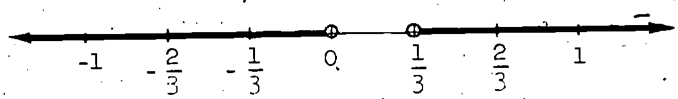
$$\frac{2}{x} < \frac{1}{x} + 3 \text{ and } x < 0 \iff \frac{1}{x} < 3 \text{ and } x < 0$$
$$\iff 1 > 3x \text{ and } x < 0$$
$$\iff \frac{1}{3} > x \text{ and } x < 0$$

Solution set: $\{x : x < 0\}$

The solution set of $\frac{2}{x} < \frac{1}{x} + 3$ and $x \neq 0$ is

$$\{x : x > \tfrac{1}{3}\} \cup \{x : x < 0\} = \{x : x < 0 \text{ or } x > \tfrac{1}{3}\}.$$

Its graph is:



Exercises 23-4

1. For each of the following expressions, state the set of real numbers for which the number represented by the expression is zero, is negative, is positive.

Expression	0	Negative	Positive
(a) $x - 2$	$\{2\}$	$\{x : x < 2\}$	______
(b) $x + 2$			
(c) $x^2 - 9$			
(d) $x^2 + 9$			
(e) $-3x$			
(f) $-(x^2 + 1)$			
(g) $\lvert x\rvert - 1$			
(h) $\lvert x\rvert + 1$			

Find the solution set for each inequality ~~and~~ draw number line graphs as indicated.

2. (a) $\frac{y}{y-2} < 3$ and $y > 2$

(b) $\frac{y}{y-2} < 3$ and $y < 2$

(c) $\frac{y}{y-2} < 3$ and $y \neq 2$; draw the graph.

3. (a) $\frac{2}{x} - \frac{3}{x} < 5$ and $x > 0$

(b) $\frac{2}{x} - \frac{3}{x} < 5$ and $x < 0$

(c) $\frac{2}{x} - \frac{3}{x} < 5$ and $x \neq 0$; draw the graph

(d) $\frac{x}{2} - \frac{x}{3} < 5$

4. (a) $\frac{2}{x} + 3 > \frac{1}{x}$ and $x > 0$

(b) $\frac{2}{x} + 3 > \frac{1}{x}$ and $x < 0$

(c) $\frac{2}{x} + 3 > \frac{1}{x}$ and $x \neq 0$; draw the graph.

5. (a) $\frac{y}{y+3} > 2$ and $y > -3$

(b) $\frac{y}{y+3} > 2$ and $y < -3$

(c) $\frac{y}{y+3} > 2$ and $y \neq -3$

For each of the following problems (a) write a mathematical sentence which is a suitable model for the situation, (b) find the solution set of the sentence, and (c) answer the question in the problem.

6. In planning a school building, it is decided that, in order to allow enough air, each room should contain at least 270 cubic feet for each pupil. A room 30 feet by 24 feet is to seat 36 pupils. At what height might the ceiling be placed?

7. An electrical circuit consists of two resistances connected in parallel. The larger resistance is twice the smaller resistance, and the total resistance is less than 2 ohms. What is the smaller resistance?

8. Mr. Brown drives 4 miles through traffic at an average rate of 20 miles per hour. What should be his average rate for 36 miles of freeway driving if he is to cover the entire 40 miles in less than 45 minutes?

9. The disinfectant on hand in a certain hospital contains 30% active ingredient. To reduce the proportion of active ingredient, water can be added. How much water would you add to 2 quarts of the disinfectant so that the proportion of active ingredient would be more than 10% but would not exceed 20%?

23-5. Equations Involving Factors Whose Product is 0

Exercises 23-5a

(Class Discussion)

Suppose that you want to solve the equation

$$x^2 - 2x = 0.$$

1. (a) Using the addition property of equality, an equivalent equation is

$x^2 =$ ________ .

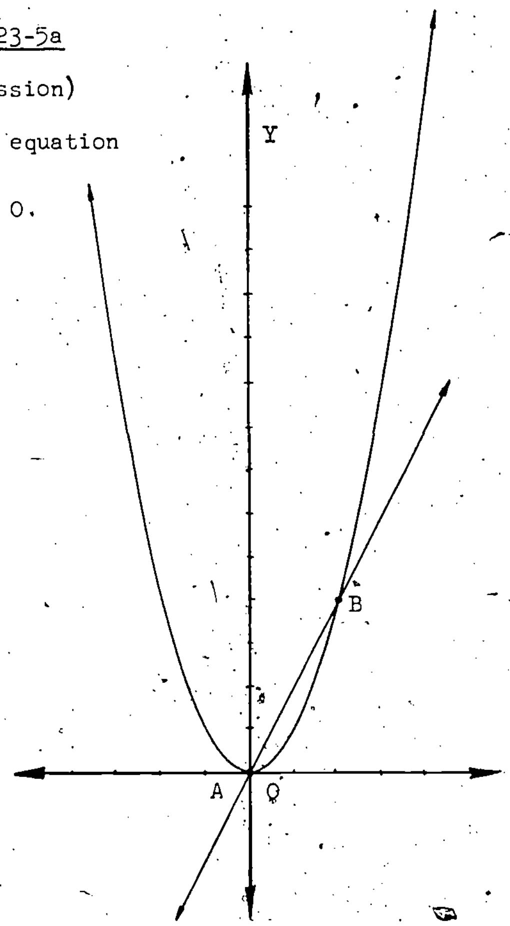
(b) Study this figure showing the graphs of the two functions

$f : x \rightarrow x^2$

and

$g : x \rightarrow 2x.$

For $x^2 = 2x$ to be true, we are looking for values of x for which the outputs of the two functions are the same. What points on the graphs fulfill this requirement?



52

(c) What are the coordinates of the points of intersection? Then for what values of x does $x^2 = 2x$? What is the solution set of $x^2 - 2x = 0$?

2. (a) For the equation $x^2 - 2x = 0$, we can use the distributive property to write the equivalent equation $x(\_\_\_\_\_\_) = \_\_\_\_\_$.

(b) Verify that the members of the solution set as found graphically also satisfy the equation

$$x(x - 2) = 0.$$

From the equation

$$x(x - 2) = 0$$

we might guess that the solution set is $\{0, 2\}$. Our guess might be based on our knowledge that for any two numbers a and b, if either or both of them are 0, then the product is 0. Actually, what we need here is the reverse statement: <u>if</u> the product of two numbers is 0, then at least one of them must be 0.

Exercises 23-5b

(Class Discussion)

1. Supply reasons for the steps of the following proof of the statement: For all real numbers a and b, if $ab = 0$, then $a = 0$ or $b = 0$.

For any number, a, $a = 0$ or $a \neq 0$, but not both. If $a = 0$, then "$a = 0$ or $b = 0$" is true and the theorem is true.

If $a \neq 0$, then a has a reciprocal $\frac{1}{a}$. If $ab = 0$ and $a \neq 0$, then

(a) $\frac{1}{a}(ab) = \frac{1}{a} \cdot 0$ why?

(b) $\frac{1}{a}(ab) = 0$ why?

(c) $(\frac{1}{a} \cdot a)b = 0$ why?

(d) $1 \cdot b = 0$ why?

(e) $b = 0$ why?

The statement just proved and the multiplication property of 0 can be combined into a single theorem:

For all real numbers, a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

What we have proved is an important fact that will be helpful in solving equations involving products. Here are some examples.

Example 1. Solve $2x^2 + 5x = 0$.

$2x^2 + 5x = 0 \iff x(2x + 5) = 0,$		Distributive property.
$\iff x = 0$ or $2x + 5 = 0.$		If $ab = 0$, then $a = 0$ or $b = 0$.
$\iff x = 0$ or $2x = -5$		Addition property of equality.
$\iff x = 0$ or $x = -\frac{5}{2}$		Multiplication property of equality.

The solution set of the last sentence, and hence of the original sentence, is $\{0, -\frac{5}{2}\}$.

Example 2. Solve $x^2 - 25 = 0$.

$x^2 - 25 = 0 \iff (x - 5)(x + 5) = 0,$		$a^2 - b^2 = (a + b)(a - b)$
$\iff x - 5 = 0$ or $x + 5 = 0.$		If $ab = 0$, then $a = 0$ or $b = 0$.
$\iff x = 5$ or $x = -5$		Addition property of equality.

The solution set of each sentence in the chain is $\{-5, 5\}$.

Example 3. Solve $x(x + 3) = 2(x + 3)$.

$x(x + 3) = 2(x + 3) \iff x(x + 3) - 2(x + 3) = 0,$		Addition property of equality.
$\iff (x - 2)(x + 3) = 0,$		Distributive property.
$\iff x - 2 = 0$ or $x + 3 = 0.$		If $ab = 0$, then $a = 0$ or $b = 0$.
$\iff x = 2$ or $x = -3.$		Addition property of equality.

The solution set is $\{2, -3\}$.

In Example 3, note that we cannot simply multiply each side of the original equation by $\frac{1}{x + 3}$ and get $x = 2$ as an equivalent equation. Why?

Exercises 23-5c

Solve each of the following equations.

1. $(x + 2)(x - 5) = 0$
2. $(x - 4)^2 = 0$
3. $x^2 - 4 = 0$
4. $x^2 - 4x = 0$
5. $x^2 - 2 = 0$ (this is equivalent to $x^2 - (\sqrt{2})^2 = 0$)
6. $x^2 + 2 = 0$
7. $x(x - 7) = 3(x - 7)$
8. $5(x + 1) = x(x + 1)$
9. $(x + 3)(2x + 1)(4x - 3) = 0$
10. $x^3 = 25x$

Another use of the equivalence

$$ab = 0 \iff a = 0 \text{ or } b = 0$$

is in solving a fractional equation on such as

$$\frac{x - 3}{x + 1} = 0 \text{ and } x \neq -1.$$

Exercises 23-5d

(Class Discussion)

1. $\frac{x - 3}{x + 1} = 0$ and $x \neq -1 \iff (x - 3)(\_\_\_\_) = 0$ and $x \neq -1$

$\iff (\_\_\_\_ = 0$ or $\frac{1}{x + 1} = 0)$ and $x \neq -1$

$\iff (\_\_\_\_$ or $\frac{1}{x + 1} = 0)$ and $x \neq -1$

2. There is no real number x such that $\frac{1}{x + 1} = 0$; hence the solution set is ______.

3. An important fact to be noted here is that if a fraction has the value 0, then the ________ of the fraction is equal to 0. The fraction has no meaning if the value of the denominator is ________.

In general terms, we can state that, for all real numbers a, and $b \neq 0$, $\frac{a}{b} = 0$ if and only if $a = 0$.

Exercises 23-5e

Solve:

1. (a) $\frac{x^2}{x^2 + 1} = 0$

   (b) $\frac{x^2}{x^2 + 1} = \frac{1}{x^2 + 1}$

   (c) $\frac{x^2 + 1}{x^2 + 1} = 0$

2. (a) $\frac{4}{3} - \frac{y}{5} - \frac{1}{2} = 0$ (Suggestion: write $\frac{4}{3} - \frac{y}{5} - \frac{1}{2}$ as a single fraction.)

   (b) $\frac{3}{4} - \frac{5}{y} - 2 = 0$ and $y \neq 0$

3. (a) $(x - 5)(x^2 - 16) = 0$

   (b) $(x^2 - 5)(x^2 - 16) = 0$

   (c) $(x^2 + 5)(x^2 - 16) = 0$

4. (a) $\frac{2a - 5}{7} - \frac{a + 5}{5} = 0$

   (b) $\frac{a + 5}{2a - 5} - \frac{5}{7} = 0$ and $a \neq \frac{5}{2}$

   (c) $\frac{5}{a + 5} - \frac{7}{2a - 5} = 0$ and $a \neq -5$, $a \neq \frac{5}{2}$

23-6. Inequalities Involving Products

We have seen that, for all real numbers a and b, $ab = 0 \Longleftrightarrow a = 0$ or $b = 0$. Now let us look at some inequalities concerning a, b, and 0.

Exercises 23-6a

(Class Discussion)

1. Explain how you know that, if a and b are real numbers such that $ab \neq 0$, then neither a nor b has the value 0.

2. If $ab \neq 0$, then either $ab > 0$ or ab ____ 0.

3. If $ab > 0$, then we are sure that either $a > 0$ and b ____ 0; or a ____ 0 and b ____ 0.

4. If $ab < 0$, what do we know about a and b?

5. If $\frac{a}{b} > 0$ and $b \neq 0$, what do we know about a and b?

6. If $\frac{a}{b} < 0$ and $b \neq 0$, what do we know about a and b?

Obviously, the solution set of a sentence in any of the forms stated in Exercises 3 to 6 above is more complicated to determine than that of an equation of the form $ab = 0$, or the form $\frac{a}{b} = 0$. However, no new properties are involved.

Example 1. Solve $(x + 2)(x - 3) > 0$.

We know that, for all real numbers a and b, if the product ab is positive, then either both factors are positive or both factors are negative. In symbols, this can be stated

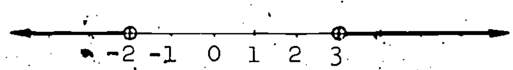
$$ab > 0 \Longleftrightarrow (a > 0 \text{ and } b > 0) \text{ } \underline{\text{or}} \text{ } (a < 0 \text{ and } b < 0).$$

Thus

$$\begin{aligned}(x + 2)(x - 3) > 0 &\Longleftrightarrow (x + 2 > 0 \text{ and } x - 3 > 0) \text{ } \underline{\text{or}} \text{ } (x + 2 < 0 \text{ and } x - 3 < 0)\\ &\Longleftrightarrow (x > -2 \text{ and } x > 3) \text{ } \underline{\text{or}} \text{ } (x < -2 \text{ and } x < 3)\\ &\Longleftrightarrow x > 3 \text{ } \underline{\text{or}} \text{ } x < -2.\end{aligned}$$

The solution set is $\{x : x < -2 \text{ or } x > 3\}$.

The number line graph of this set is



Example 2. Solve $(x + 2)(x - 3) < 0$.

For all real numbers a and b, if the product ab is negative, then one of the factors is positive and the other is negative. In symbols,

$$ab < 0 \Longleftrightarrow (a > 0 \text{ and } b < 0) \text{ } \underline{\text{or}} \text{ } (a < 0 \text{ and } b > 0).$$

Hence $(x + 2)(x - 3) < 0 \iff (x + 2 > 0 \text{ and } x - 3 < 0)$ <u>or</u>

$$(x + 2 < 0 \text{ and } x - 3 > 0)$$

$$\iff (x > -2 \text{ and } x < 3) \text{ \underline{or} } (x < -2 \text{ and } x > 3).$$

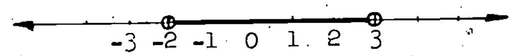
Since there is <u>no</u> number which is <u>both</u> less than -2 <u>and</u> greater than 3, the solution set of the second part is $\emptyset$, and we have the equivalence.

$$(x + 2)(x + 3) < 0 \iff x > -2 \text{ and } x < 3$$

$$\iff -2 < x < 3$$

The solution set is $\{x : -2 < x < 3\}$.

The number line graph is



<u>Example 3</u>. Solve $\frac{x + 2}{x - 3} > 0$ and $x \neq 3$.

For all real numbers a and b, if the quotient $\frac{a}{b}$ is positive, then either both a and b are positive or both a and b are negative. In symbols

$$\frac{a}{b} > 0 \iff a > 0 \text{ and } b > 0, \text{ \underline{or} } a < 0 \text{ and } b < 0$$

Hence $\frac{x + 2}{x - 3} > 0$ and $x \neq 3 \iff (x + 2 > 0$ and $x - 3 > 0$ and $x \neq 3)$

<u>or</u> $(x + 2 < 0$ and $x - 3 < 0$ and $x \neq 3)$.

The solution set and the graph are the same as those in <u>Example 1</u> above.

<u>Exercises 23-6b</u>

1. Solve $(2x + 3(x - 3) > 0$ and graph its solution set.

2. Solve $x^2 < 25$ and graph its solution set.

3. Solve $x^2 + 3x < 0$ and graph its solution set.

4. Solve $\frac{3x - 2}{x + 4} > 0$ and $x \neq -4$ and graph its solution set.

5. (a) Solve each of these: $(x + 2)(x - 3) = 0$

   $(x + 2)(x - 3) < 0$

   $(x + 2)(x - 3) > 0$

   (b) Draw the graphs of the three sentences on three parallel number lines. What do you observe about the union of the three sets?

6. (a) Solve each of these: $\frac{x - 2}{x + 3} = 0$ and $x \neq -3$

$$\frac{x - 2}{x + 3} < 0 \text{ and } x \neq -3$$

$$\frac{x - 2}{x + 3} > 0 \text{ and } x \neq -3$$

(b) Draw their graphs on three parallel number lines. What do you observe about the union of the three sets?

Suppose that you needed to find the solution set of the inequality

$$(x + 3)(x + 2)(x - 1) > 0.$$

To use a compound sentence for the purpose is very complicated, since for a product abc of real numbers to be positive we have to consider all possible cases in which two factors are negative, as well as the case in which <u>no</u> factor is negative. In symbols,

$$\begin{aligned} abc > 0 \iff\ & a > 0 \text{ and } b > 0 \text{ and } c > 0, \ \underline{\text{or}} \\ & a > 0 \text{ and } b < 0 \text{ and } c < 0, \ \underline{\text{or}} \\ & a < 0 \text{ and } b > 0 \text{ and } c < 0, \ \underline{\text{or}} \\ & a < 0 \text{ and } b < 0 \text{ and } c > 0. \end{aligned}$$

The following use of a number line graph shows a more effective way for dealing with such inequalities.

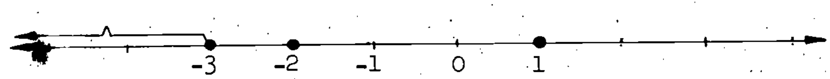
<u>Exercises 23-6c</u>

(Class Discussion)

1. Solve $(x + 3)(x + 2)(x - 1) = 0$, and show its graph on the number line.

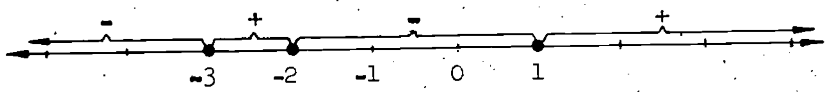
2. What is true for each of the factors $(x + 3)$, $(x + 2)$ and $(x - 1)$ for any x less than -3? (Try $x = -4$, for example.)

3. We indicate on the number line the fact that for any x less than -3 all three factors are negative numbers and therefore their product is negative:



Consider values of x between -3 and -2, between -2 and 1, and larger than 1 and decide for each case whether the product of the factors is positive or negative. Indicate your decision on the number line you drew for Exercise 1.

4. Since the values -3, -2, and 1 are those for which the product has the value 0, they are the points at which a factor changes its sign from negative to positive or from positive to negative. Each time that a <u>single</u> factor changes sign, the product changes sign. Check your final diagram with that shown here:



Use the diagram to help you state the solution set of each of the following and to draw the graph of each.

(a) $(x + 3)(x + 2)(x - 1) > 0$

(b) $(x + 3)(x + 2)(x - 1) < 1$

Thus the number line can be used as an aid in finding the solution set of a sentence which states that the product $abc \ldots \neq 0$ as follows:

(1) Find all values of the variable for which the product $a \cdot b \cdot c \cdot \ldots = 0$. Locate on the number line the points corresponding to such numbers.

(2) From each section into which the points separate the number line, select a single point, and determine whether its coordinate makes the product $a \cdot b \cdot c \cdot \ldots$ positive or negative.

Since the points for which the product is 0 are the only points at which a factor changes its sign from positive to negative, all points in a given section represent numbers with the same sign as the coordinate of the sample point from that section. Thus the solution sets of $a \cdot b \cdot c \cdot \ldots > 0$ and of $a \cdot b \cdot c \cdot \ldots < 0$ can be readily determined.

<u>Exercises 23-6d</u>

For each of the following, draw the graphs of the three sentences on three parallel lines, and state the solution set of each sentence.

1. (a) $x(x - 2)(x + 3) = 0$
   (b) $x(x - 2)(x + 3) > 0$
   (c) $x(x - 2)(x + 3) < 0$

2. (a) $(x + 3)(x + 1)(x - 2)(x) = 0$

(b) $(x + 3)(x + 1)(x - 2)(x) > 0$

(c) $(x + 3)(x + 1)(x - 2)(x) < 0$

3. (a) $(x^2 - 1)(x + 3) = 0$

(b) $(x^2 - 1)(x + 3) > 0$

(c) $(x^2 - 1)(x + 3) < 0$

4. (a) $(x + 5)(x + 4)(x + 2)(x)(x - 3) = 0$

(b) $(x + 5)(x + 4)(x + 2)(x)(x - 3) > 0$

(c) $(x + 5)(x + 4)(x + 2)(x)(x - 3) < 0$

5. (a) $(x - 1)^2(x + 3) = 0$

(b) $(x - 1)^2(x + 3) > 0$

(c) $(x - 1)^2(x + 3) < 0$

23-7. Summary

Section 23-1.

The replacement set for an equation in one variable is the same as the intersection of the domains of all functions involved in the equation.

Each value of the variable for which a sentence is true is called a solution of the sentence and the set of all such values is the solution set of the sentence.

Equations which have the same replacement set and the same solution set are called equivalent equations.

An equation may be solved by writing a chain of equivalent equations such that the last equation in the chain has an obvious solution set. Since the equations are all equivalent, the solution set of the final equation is the solution set of the original equation.

Proper use of any of the field properties, or of the addition or multiplication property of equality, will lead from one equation to an equivalent one.

The "properties" of equality are:

(1) Addition property of equality: For all real numbers a, b, and c, $a = b$ if and only if $a + c = b + c$.

(2) Multiplication property of equality: For all real numbers a, b, and c such that $c \neq 0$, $a = b$ if and only if $ac = bc$.

We use the symbol "$\Longleftrightarrow$" to mean "is equivalent to."

Section 23-2.

Equivalent inequalities are inequalities which have the same replacement set and the same solution set.

Proper use of any of the field properties or of the Addition or the Multiplication Property of Order enable us to write a chain of equivalent inequalities.

Stated in terms of the order relation "<", we have:

Addition Property of Order: For all real numbers a, b, and c, $a < b$ if and only if $a + c < b + c$.

Multiplication Property of Order: For all real numbers a, b, and c such that $c \neq 0$.

(1) $a < b$ if and only if $0 < c$ and $ac < bc$.

(2) $a < b$ if and only if $c < 0$ and $bc < ac$.

Since "$b > a$" can be equivalently expressed as "$a < b$", similar statements can be made in terms of the relation ">".

We used the Multiplication Property of Order to prove the following theorem about the order of the reciprocals of two numbers:

For any two nonzero real numbers a and b, if $a < b$, then

$\frac{1}{b} < \frac{1}{a}$ if both a and b are positive or both a and b are negative,

$\frac{1}{a} < \frac{1}{b}$ if a is negative and b is positive.

Section 23-3.

An equation involving fractions whose denominators have the value 0 for some values of the variable should be rewritten as a compound sentence by adding a clause or clauses indicating any restriction on the replacement set.

For example

$$\frac{3}{x} + \frac{5}{x - 1} = \frac{2}{x}$$

has no meaning for $x = 0$ or $x = 1$, hence we write

$$\frac{3}{x} + \frac{5}{x - 1} = \frac{2}{x} \text{ and } x \neq 0, \quad x \neq 1.$$

For such compound sentences, equivalent sentences can be written by applying any of the field properties or the addition or multiplication property of equality.

A fractional equation of the form

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$$

was found to have several applications, including problems concerned with filling a tank, with an electric circuit made up of two resistances connected in parallel, and with the object and image distances in relation to a lens.

Section 23-4.

In applying the Multiplication Property of Order to write a chain of inequalities equivalent to a fractional inequality in which the multiplier contains the variable, separate consideration must be given to each case -- that in which the multiplier is positive and that in which it is negative.

Section 23-5.

A theorem which enables us to solve an equation for which we can write an equivalent equation in the form $ab = 0$ is:

For all real numbers, a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

Another theorem, which follows from the one just stated, is:

For all real numbers a, and $b \neq 0$, $\frac{a}{b} = 0$ if and only if $a = 0$.

Section 23-6.

For all real numbers a and b, $ab > 0$ if and only if: $a > 0$ and $b > 0$, <u>or</u> $a < 0$ and $b < 0$

For all real numbers a and b, $ab < 0$ if and only if: $a > 0$ and $b < 0$, <u>or</u> $a < 0$ and $b > 0$

For real numbers a and b, $b \neq 0$, $\frac{a}{b} > 0$ if and only if: $a > 0$ and $b > 0$, <u>or</u> $a < 0$ and $b < 0$

For real numbers a and b, $b \neq 0$, $\frac{a}{b} < 0$ if and only if: $a > 0$ and $b < 0$, <u>or</u> $a < 0$ and $b > 0$

If a sentence states that the product of three or more factors is greater than or less than 0, then the solution set of the sentence by means of an equivalent compound sentences becomes very complicated. An alternative method uses a number line diagram as follows:

(1) Find all values of the variable for which the product is 0. Locate on the number line the points corresponding to such numbers.

(2) From each section into which the points separate the number line select a single point and determine whether its coordinate makes the product positive or negative.

(3) Since all points in a given section have the same effect on the sign of the product, the solution set of either $a \cdot b \cdot c \cdot \ldots > 0$ or of $a \cdot b \cdot c \cdot \ldots < 0$ can be determined easily from the number line.

QUADRATIC FUNCTIONS

24-1. Introduction

Consider the function

$$f : x \to ax^2 + bx + c$$

where a, b, and c are real numbers.

If $a = 0$, the function becomes

$$g : x \to bx + c.$$

Its graph is a non-vertical line. Figure 1 shows the graph in the XY-plane of the function

$$g : x \to 3x - 2.$$

(Recall that the equation $y = bx + c$ describes the output y of the function $g : x \to bx + c$.)

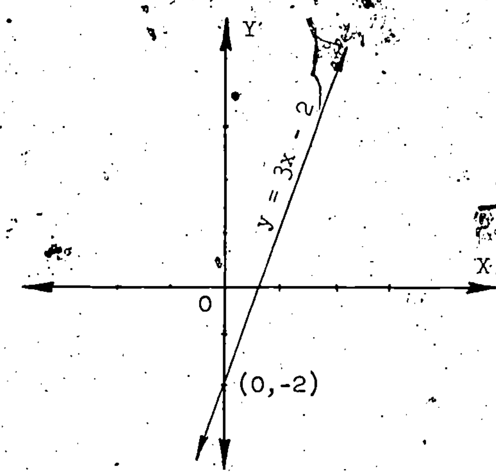


Figure 1

In particular, if b also is 0, the function becomes the constant function

$$h : x \to c.$$

Its graph is a horizontal line. In Figure 2 we show the graph of

$$h : x \to -3.$$

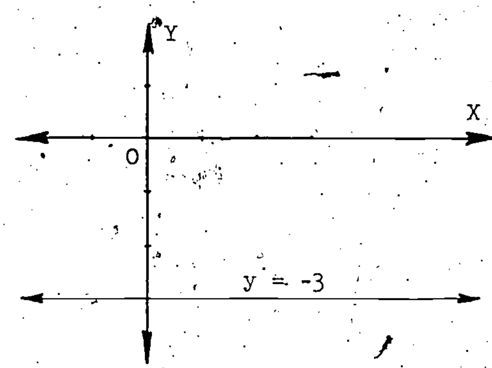


Figure 2

If $a \neq 0$, the function

$$f : x \to ax^2 + bx + c$$

is called a quadratic function. The word "quadratic" is derived from the Latin verb "quadrare", which means "to make square"; the expression ax^2

involves the square of the variable x. Similarly, for $a \neq 0$, an expression of the type $ax^2 + bx + c$ is called a quadratic expression. An equation of the type $ax^2 + bc + c = 0$, $a \neq 0$, is called a quadratic equation.

Figure 3 shows the graph of the basic quadratic function

$$f : x \rightarrow x^2.$$

The graph of any function of the type

$$x \rightarrow ax^2 + bx + c, \ a \neq 0$$

has this general shape, and is called a parabola.

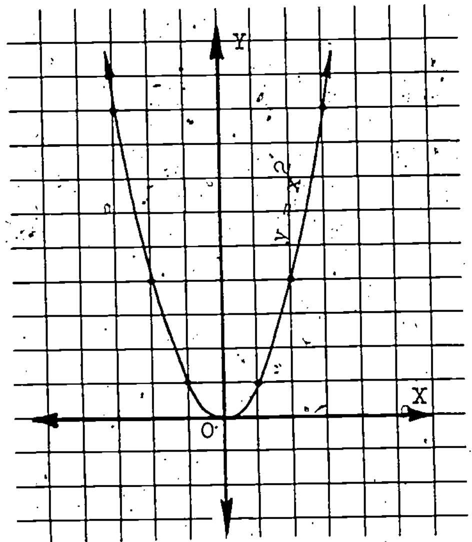


Figure 3

Exercises 24-1

(Class Discussion)

1. Consider the function

$$f : x \rightarrow ax^2 + bx + c.$$

Name which type of function, constant, linear, or quadratic, f is, given that :

(a) $a = 0, \ b = 0, \ c \neq 0$

(b) $a = 0, \ b \neq 0, \ c = 0$

(c) $a = 0, \ b \neq 0, \ c \neq 0$

(d) $a \neq 0, \ b = 0, \ c = 0$

(e) $a \neq 0, \ b \neq 0, \ c = 0$

(f) $a \neq 0, \ b \neq 0, \ c \neq 0$

2. About the year 1590, Galileo proved that the velocity of a freely falling object is directly proportional to the time it falls. To express this as a function of time t we might write

$$h : t \rightarrow bt.$$

(a) Find the value of b in the function

$$h : t \to bt$$

if you observe that in 2 seconds a ball dropped from the top of a building attains a speed of 64 ft/sec.

(b) Use the value for b found in part (a) to write the function relating time in seconds to speed in ft/sec. Is it a constant, a linear, or a quadratic function?

(c) What is the value of the output of

$$h : t \to 32t$$

for an input of 5? If a ball is dropped from a high building, how fast is it falling at the end of 5 seconds?

3. Some equations with which we have dealt in earlier chapters are:

$x^2 = 8001$ $\quad 4t - t^2 = 3$ $\quad x^2 + x = 1$

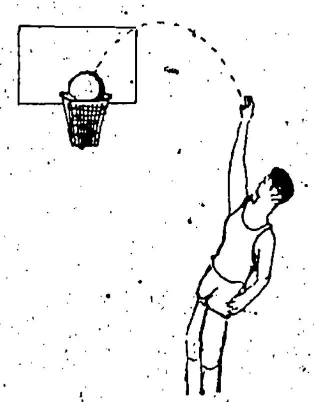
$x^2 - 2x = 0$ $\quad 64t - 16t^2 = 48$

Each of these is an example of a ____________ equation.

The graph of each is a ____________.

____

The parabola is a curve which occurs in many physical situations. It is the path of a basketball as it is tossed into the basket, the path of a baseball hit for a home run, the path of a bullet shot from a gun. Water falling over a precipice follows a parabolic path, as do sparks from a skyrocket in a fireworks display.



When a parabola is rotated about its axis, it traces a parabolic surface, called a <u>paraboloid</u>. When a source of light is placed at a certain point, called the focus, all of the light striking the surface is reflected

in parallel lines, as shown in Figure 4. The headlights of cars, searchlights, and beamed radio transmissions make use of this property.

In a similar fashion, a parabolic surface can be used to collect heat from sunlight, by concentrating at the focus the parallel rays from the sun. This use of a paraboloid as an "accumulator" also occurs in telescopes and in radar listening devices.

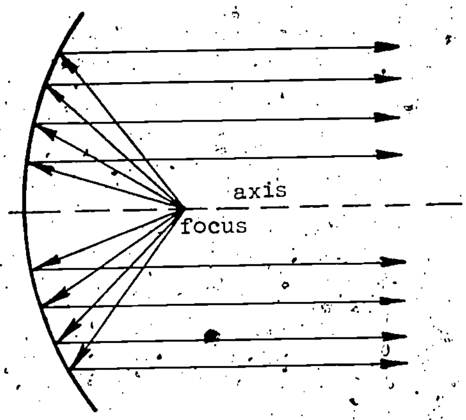


Figure 4

24-2. Functions of the Type $x \to ax^2$, $a \neq 0$

In the last section we defined a quadratic function as a function of the general form

$$f : x \to ax^2 + bx + c, \quad \text{and} \quad a \neq 0.$$

In this section we shall look at the special case of the quadratic function for which $b = 0$ and $c = 0$.

We called the graph in Figure 3 of Section 24-1 the graph of the basic quadratic function

$$f : x \to x^2;$$

that is, the graph of the function

$$x \to ax^2 \quad \text{for} \quad a = 1.$$

Figure 5 shows the graphs of the equations $y = x^2$ and $y = |x|$ on the same set of axes.

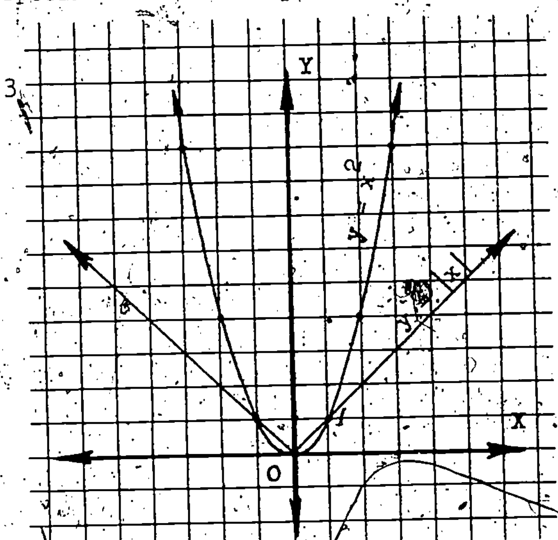


Figure 5

Exercises 24-2a

(Class Discussion)

Use Figure 5 to help you to answer the following questions.

1. For what set of ordered pairs is it true that $y = x^2$ and $y = |x|$?

2. (a) For what set of values of x is $|x| > x^2$?

   (b) For what set of values of x is $|x| < x^2$?

3. (a) For $\{x : 0 < x < 1\}$ is the graph of $y = x^2$ above or below the graph of $y = |x|$? For what other set of values of x is the same fact true of the two graphs?

   (b) For $\{x : x < -1\}$ is the graph of $y = x^2$ above or below the graph of $y = |x|$? For what other set of values of x is the same fact true of the two graphs?

4. What is the slope of

   (a) the ray which is the graph of $y = |x|$ and $x \geq 0$?

   (b) the graph of $y = |x|$ and $x \leq 0$?

Next we compare the graphs of $y = ax^2$ and of $y = a|x|$ for some cases where $a > 0$ and $a \neq 1$. Figure 6 shows the graphs of

$y = 2x^2$, $y = 2|x|$,

$y = \frac{1}{4}x^2$, $y = \frac{1}{4}|x|$.

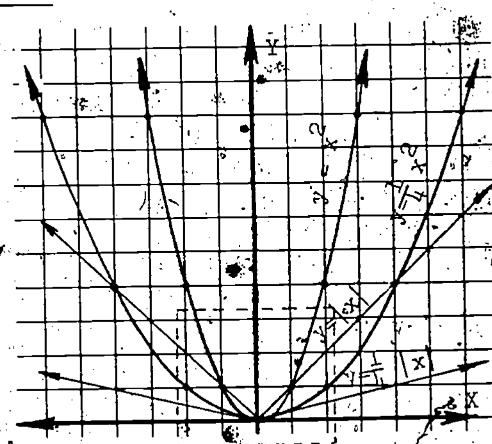
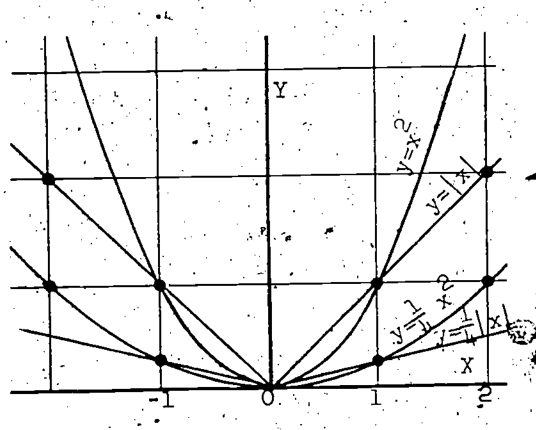


Figure 6



Exercises 24-2b.

(Class Discussion)

Use Figure 6 to help you answer the following questions.

1. For what set of ordered pairs is it true that

   (a) $y = 2x^2$ and $y = 2|x|$?

   (b) $y = \frac{1}{4}x^2$ and $y = \frac{1}{4}|x|$?

2. For what set of values of x is

   (a) $2x^2 < 2|x|$? (b) $2|x| < 2x^2$?

   (c) $\frac{1}{4}x^2 < \frac{1}{4}|x|$? (d) $\frac{1}{4}|x| < \frac{1}{4}x^2$?

3. (a) For all nonzero values of x, is the graph of $y = 2x^2$ above or below the graph of $y = x^2$? Is the graph of $y = \frac{1}{4}x^2$ above or below the graph of $y = x^2$?

   (b) For all nonzero values of x, is the graph of $y = \frac{1}{4}|x|$ above or below the graph of $y = |x|$? Is the graph of $y = 2|x|$ above or below the graph of $y = |x|$?

4. Let us see what general observations we can make concerning $y = ax^2$ for $a > 0$.

   (a) If $x = 0$, then $y =$ ___; if $|x| = 1$, then $y = a$; if $0 < |x| < 1$, then $0 < y <$ ___; if $|x| > 1$, then $y >$ ___.

   (b) The graph of $y = ax^2$, $a > 0$, is a parabola which opens upward. Its lowest point is the origin. If $a = 1$, then $y = x^2$, and its graph is the parabola shown in Figure 3.

      (1) If $0 < a < 1$, then does the graph of $y = ax^2$ lie above or below the graph of $y = x^2$, except at $(0, 0)$?

      (2) If $a > 1$, then except at $(0,0)$ does the graph of $y = ax^2$ lie above or below the graph of $y = x^2$?

   (c) For all real values of x, which of the following is the range of y? $A = \{y : y > 0\}$, $B = \{y : y \geq 0\}$; $C = \{y : y$ is a real number$\}$.

5. Think about the graph of the equation $y = ax^2$ for $a < 0$ and try to answer these questions.

(a) Does the origin lie on the graph?

(b) Does the graph contain any points for which the ordinate (i.e., the value of y) is positive?

(c) Describe how you think the graph of the first equation in each pair compares with the graph of the second equation.

$y = -x;\ y = x$

$y = -3x;\ y = 3x$

$y = -\frac{1}{2}x;\ y = \frac{1}{2}x$

$y = 5x;\ y = -5x$

In summary, the graph of the function

$$f : x \rightarrow ax^2,\ a > 0$$

is a <u>parabola</u> which contains the origin. The <u>domain</u> of the function is the set of all real numbers; its <u>range</u> is the set of all non-negative real numbers. Thus the origin is the lowest point on the graph and the ordinate, 0, of the origin is the minimum (or least) value of the function.

Let us think of the graph of

$$x \rightarrow ax^2,\ a = 1,$$

shown here, as the "basic" parabola. It is apparent that if $a > 1$, then for any nonzero input the output of the function is greater than the corresponding output for the function whose graph is the basic parabola. Hence the graph of

$$x \rightarrow ax^2,\ a > 1$$

is a parabola for which the origin is the minimum, or lowest, point, but which is narrower and steeper than the basic parabola.

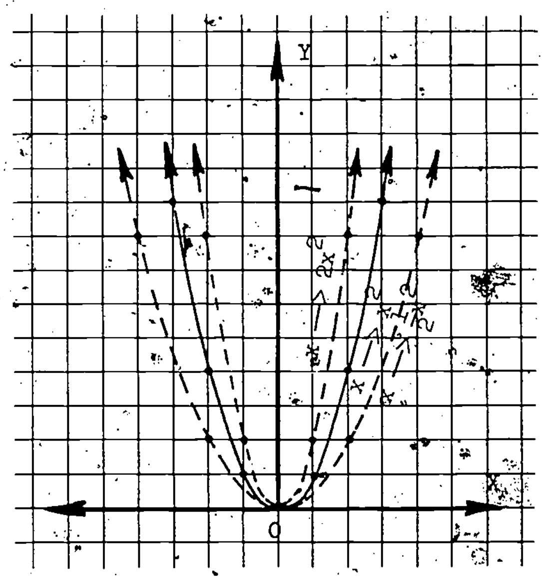


Figure 7

On the other hand, if $0 < a < 1$, then for each nonzero input of the function

$$x \to ax^2, \quad 0 < a < 1$$

the output is <u>less</u> than the corresponding output of the function

$$x \to x^2$$

and the graph is flatter and broader than the basic parabola.

In the final exercise in Exercises 24-2b you were asked to think about the graph of $y = ax^2$ for $a < 0$. Another way to describe the graph is to call it the graph of the function

$$x \to -ax^2, \quad a > 0.$$

Compare this with the function

$$x \to ax^2, \quad a > 0$$

and note the following facts about their graphs.

(1) For any nonzero value of a, the graphs of the two functions have the same shape.

(2) For each value of x, the ordinate of the point on the graph of one function is the <u>opposite</u> of the ordinate of the corresponding point on the graph of the other function.

(3) The graph of $x \to -ax^2$, $a \neq 0$, is the graph of $x \to ax^2$, $a \neq 0$, reflected in the x-axis.

(4) The <u>domain</u> of the function

$$x \to ax^2, \quad a < 0$$

is the set of all real numbers; the <u>range</u> is $\{y : y \leq 0\}$.

Exercises 24-2c

1. On one set of axes, draw the graphs of the following functions.

   (a) $x \to |x|$ (b) $x \to x^2$

   (c) $x \to -|x|$ (d) $x \to -x^2$

2. On one set of axes, draw the graphs of the following functions.

   (a) $x \to \frac{1}{3}|x|$ (b) $x \to \frac{1}{3}x^2$

   (c) $x \to -\frac{1}{3}|x|$ (d) $x \to -\frac{1}{3}x^2$

3. If $a = 0$ in the function $x \to ax^2$, what kind of function results? What is its graph?

4. Draw the graph of the function

$$f : x \to \frac{1}{2} x^2.$$

(a) For these points on the graph of f, what are their images for the reflection on the Y-axis?

point	image
$(2,2)$	____
$(-3,\frac{9}{2})$	____
(u,v)	____

(b) Are the images of the points also on the graph?

(c) Fold your graph paper along the Y-axis; what do you observe about the two parts of the parabola?

(d) Recall that when a figure is invariant for a reflection on a line ℓ, the figure is said to be symmetric, with line ℓ as axis of symmetry. What is the axis of symmetry of the parabola that is the graph of f?

5. For each of the following functions, tell whether or not the graph is symmetric. If the graph is symmetric, state the axis of symmetry.

(a) $f : x \to ax^2, \quad a \neq 0$

(b) $g : x \to ax, \quad a \neq 0$

(c) $h : x \to a, \quad a \neq 0$

(d) $k : x \to a|x|, \quad a \neq 0$

6. In his experiment with falling bodies, Galileo showed that the distance traveled by a falling body is a function of the time spent in falling. When the time is t seconds and the distance is s feet, the relation is described by the equation

$$s = 16t^2,$$

which is the rule for the function

$$f : t \to 16t^2.$$

(a) Draw the graph of f. (<u>Hint</u>: instead of using the same scale on both axes, let 1 unit on the Y-axis correspond to 8 units on the X-axis.)

(b) If a ball is dropped from the top of a building 400 feet high, in how many seconds will it reach the ground? If the function

$$f : t \to 16t^2$$

is to serve as a mathematical model of the physical situation, it is apparent that the domain of the function is restricted to the set of non-negative numbers. Thus you must solve the equation

$$16t^2 = 400 \text{ and } t \geq 0.$$

$$16^2 = 400 \text{ and } t \geq 0 \iff t^2 = 25 \text{ and } t \geq 0.$$

(c) If a ball dropped from the top of a tower hits the ground in 6 seconds, how high is the tower? Again the function

$$f : t \to 16t^2$$

can serve as a mathematical model of the physical situation if $0 \leq t \leq 6$. This time you are looking for the output corresponding to an input of 6.

7. Suppose that a ball is dropped from the top of a tower and hits the ground after 10 seconds. How high is the tower?

Consider the function f such that for each input we get an output which is the square of the input. We specify this as $f : x \to x^2$. A frequently used notation for indicating the output of a function f for a given input n is "f(n)", which is usually read "<u>f</u> of <u>n</u>", or "<u>f</u> at <u>n</u>". For the function

$$f : x \to x^2$$

we can write $f(1) = 1^2 = 1$, $f(-3) = (-3)^2 = 9$, $f(n) = n^2$.

Thus: "$f : x \to f(x)$, where $f(x) = x^2$" says "the function f assigns to each input x, from the domain, an output f(x) such that f(x) is x^2". The <u>function</u> is f, the <u>input</u> is a value of x, and the <u>output</u> is the value f(x) which the function assigns to x.

For brevity we sometimes say merely

$$f(x) = x^2,$$

but the longer sentence in the previous paragraph is what we mean.

When we refer to the graph of $f(x) = x^2$

we mean the graph of $f : x \to x^2$,

which is the graph of $y = x^2$ on the XY-plane.

For this graph, any point (x,y) can be designated as (x,y), or as $(x,f(x))$, or as (x,x^2).

Exercises 24-2d

1. Among the points $(x,f(x))$ on the graph of $f : x \to 2x^2$ are

 (a) $(\_\_\_, 2)$ and $(-1,2)$

 (b) $(2,8)$ and $(\_\_\_, 8)$

 (c) $(3,18)$ and $(\_\_\_, 18)$

 (d) $(\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, \_\_\_)$

2. (a) If (p,q) is on the graph of $x \to -\frac{1}{3}x^2$, then $(-p, \_\_\_)$ is also on the graph of $x \to -\frac{1}{3}x^2$.

 (b) If $f : x \to -\frac{1}{3}x^2$ and $f(p) = q$, then $f(-p) = \_\_\_$.

3. (a) On the graph of $y = ax^2$, $a \neq 0$, describe the location of the points (p,q) and $(-p,q)$ relative to each other and to the coordinate axes.

 (b) If the coordinate plane were folded along the ___-axis, the parts of the graph of $y = ax^2$ on either side of the line whose equation is $x = 0$ would coincide; point (p,q) would coincide with point ___.

4. (a) The graph of $f : x \to ax^2$ is symmetric about the line given by the equation ___.

 (b) The graph of $y = ax^2$ is symmetric about the ___-axis.

5. For each of the following pairs of functions, the point (u,v) is on the graph of f and the point (u,w) is on the graph of g;

determine which is correct, $v \geq w$ or $v \leq w$.

(a) $f : x \to 2x^2$; $g : x \to -2x^2$

(b) $f : x \to \frac{1}{2} x^2$; $g : x \to 2x^2$

(c) $f : x \to -\frac{1}{2} x^2$; $g : x \to -2x^2$

6. Complete the following table.

If the pair	satisfies the equation	then the pair	satisfies the equation
(2,12)	$y = 3x^2$	(2,-12)	$y = -3x^2$
(3,18)	$y = ax^2$	(3, ______)	$y = -ax^2$
(1,-5)	$y = -5x^2$	(1, ______)	$y = 5x^2$
(-2,n)	$y = ax^2$	(-2, ______)	$y = -ax^2$
(-3,-m)	$y = -bx^2$	(-3, ______)	$y = bx^2$
(u,v)	$y = ax^2$	(u,-v)	______

7. On one set of axes, sketch the graphs of $y = x^2$ and of $y = -x^2$. Then fold the paper along the X-axis.

(a) What do you observe about the two graphs?

(b) Explain which of the following rigid motions take the graph of $y = ax^2$ into the graph of $y = -ax^2$.

(1) translation

(2) rotation

(3) reflection

8. For each of the following determine a so that the graph of $y = ax^2$ contains the given point.

(a) (1,1)

(b) (1,2)

(c) (2,2)

(d) (1,-1)

(e) (2,-2)

(f) (1,-2)

9. If the point (u,v) is on the graph of $y = x^2$, then

(a) the point $(u,2v)$ is on the graph of $y =$ ____;

(b) the point $(u,$ ____$)$ is on the graph of $y = \frac{1}{2}x^2$;

(c) the point $(u,$ ____$)$ is on the graph of $y = -2x^2$;

(d) the point $(u, -\frac{1}{2}v)$ is on the graph of $y =$ ____.

10. (a) What is the sum of the first three odd integers, 1, 3, and 5?

(b) What is the sum of the first four odd integers?

(c) What is the sum of the first five odd integers?

(d) What is the sum of the first seven odd integers?

(e) What is the sum of the first ten odd integers?

(f) What would you guess to be the sum of the first twenty odd integers? Check your guess by addition.

(g) What would you guess to be the sum of the first n odd integers?

11. Consider the following abbreviated table.

Input	x	the nth odd integer
Output	$f(x)$	the sum of the first n odd integers

(a) Write a function f which associates input with output in the given table.

(b) What name have we given to this type of function?

24-3. Translations of the Parabola $x \to ax^2$.

In Chapter 19 a translation is defined as a rigid motion in which the distance between any point P and its image P' is the same as the distance between any other point Q and its image Q'. In a coordinate system a translation is defined as a function. For example, the function

$$t : (x,y) \to (x - 2, y + 3)$$

describes the translation that assigns to a point (x,y) an image that is

2 units to the left and 3 units above point (x,y), as shown in Figure 8.

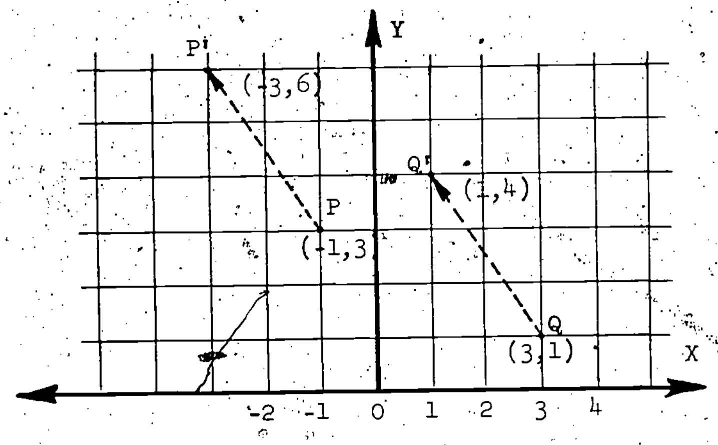


Figure 8

Exercises 24-3a

(Class Discussion)

1. A translation g is described as the function

$$g : (x,y) \to (x, y + 2).$$

(a) Find the images of the following points for translation g.

$A(-3,9) \to A'$ ______ $E(1,1) \to E'$ ______

$B(-2,4) \to B'$ ______ $F(2,4) \to F'$ ______

$C(-1,1) \to C'$ ______ $G(3,9) \to G'$ ______

$D(0,0) \to D'$ ______

(b) On one set of coordinate axes, plot and label the points and image points listed in part (a).

(c) Verify that the ordered pairs for points A-G in part (a) satisfy the equation $y = x^2$; draw the parabola which is the graph of $y = x^2$.

(d) Draw the parabola containing the image points A'-G' in (a); write an equation which is satisfied by each ordered pair for an image point in part (a).

(e) What is the axis of symmetry of the graph of $y = x^2 + a$? What is the minimum (least) value of y? What are the coordinates of the lowest point on the graph?

2. Consider the translation h described as the function

$$h: (x,y) \to (x + 3, y).$$

(a) Find the images of the following points for translation h.

$A(-3,9) \to A'$ _____	$E(1,1) \to E'$ _____
$B(-2,4) \to B'$ _____	$F(2,4) \to F'$ _____
$C(-1,1) \to C'$ _____	$G(3,9) \to G'$ _____
$D(0,0) \to D'$ _____	

(b) On one set of coordinate axes, plot and label the points and their images listed in part (a).

(c) Draw two parabolas, one through points A-G and the other through image points A'-G'.

(d) The parabola through points A-G is the graph of the equation $y =$ _____. Which of the following is an equation whose graph is the parabola through image points A'-G'?

I. $y = (x + 3)^2$ III. $y = x^2 + 3$

II. $y = (x - 3)^2$ IV. $y = x^2 - 3$

(e) What is the axis of symmetry of the graph of $y = (x - 3)^2$. What is the minimum value of y? What are the coordinates of the lowest point on the graph?

From the exercises above we can make the following observations about the graphs of the functions $x \to x^2 + 2$ and $x \to (x - 3)^2$:

(1) Each graph is a parabola that opens upward.

A. For $x \to x^2 + 2$,

(2A) The graph is the image of the graph of $x \to x^2$ under a translation of 2 units upward.

B. For $x \to (x - 3)^2$,

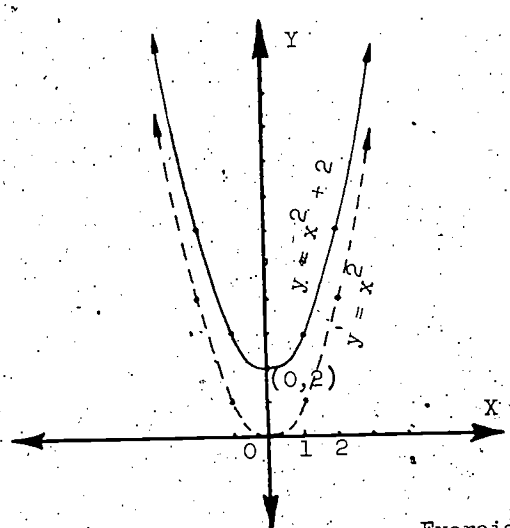
(2B) The graph is the image of the graph of $x \to x^2$, under a translation of 3 units to the right.

(3A) The axis of symmetry of the graph is the vertical line $x = 0$.

(4A) The minimum output of the function is 2.

(5A) The lowest point on the graph is $(0,2)$.

(6A) The graph is

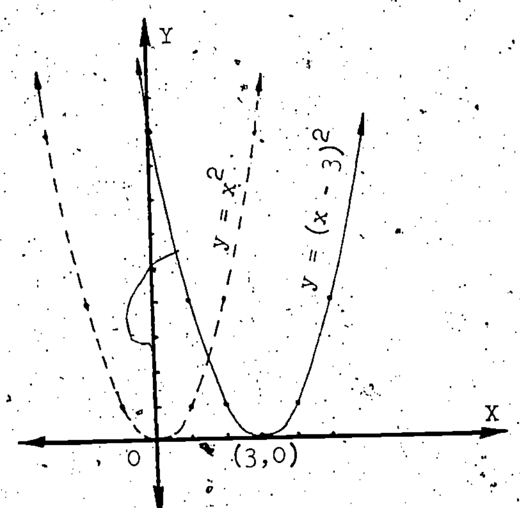


(3B) The axis of symmetry of the graph is the vertical line $x = 3$.

(4B) The minimum output of the function is 0.

(5B) The lowest point on the graph is $(3,0)$.

(6B) The graph is



Exercises 24-3b

1. For each of the following functions, make observations similar to those given above. (Consider the graph of each function to be a translation of the graph of a corresponding function of the form $x \to ax^2$, $a \neq 0$.)

(a) $x \to 2x^2 + 1$

(b) $x \to \frac{1}{2}x^2 - 3$

(c) $x \to -\frac{1}{3}x^2 - 4$

(Hint: if a parabola open downward, it has a highest point, and the function has a maximum (greatest) value.)

(d) $x \to 2(x - 1)^2$

(e) $x \to \frac{1}{2}(x + 3)^2$ (Hint: $x + 3 = [x - (-3)]$)

(f) $x \to -\frac{1}{3}(x - 4)^2$

(g) $x \to x^2 + k$, $k < 0$

(h) $x \to -a(x - h)^2$, $a > 0$, $h > 0$

2. (a) For the functions in Exercise 1, describe how, in some cases, the graph could be considered to be a reflection, followed by a translation, of the graph of a corresponding function of the form $x \to ax^2$, $a > 0$.

   (b) Could each of the graphs in Exercise 1 (c), (f), and (h) also result from a translation, followed by a reflection, of the graph of $x \to ax^2$, $a > 0$? Explain.

3. Show how your observations in Exercise 1 can help you to sketch the graphs in parts (b), (c), (e), and (f) quickly.

We have noted that each parabola has an axis of symmetry, which we shall call the <u>axis</u> of the parabola. The intersection of a parabola with its axis is called the <u>vertex</u> of the parabola. An examination of the parabolas which have been considered so far will show that if a parabola opens <u>upward</u>, then the vertex is the <u>lowest</u> point. If a parabola opens <u>downward</u>, then its vertex is its <u>highest</u> point.

Exercises 24-3c

(Class Discussion)

1. Draw the graph of the function $f : x \to x^2$. Write functions whose graphs are images of the graph of f under the following translations:

   (a) 2 units upward.

   (b) 3 units to the right.

   (c) 2 units upward, followed by 3 units to the right.

   (d) 1 unit downward.

   (e) 5 units to the left.

   (f) 5 units to the left, followed by 1 unit downward.

2. For the function $g : x \to ax^2$, $a > 0$, write functions whose graphs are images of the graph of g under the following rigid motions, where $h > 0$ and $k > 0$.

   (a) A reflection on the X-axis.

(b) A translation h units to the right.

(c) A translation k units upward.

(d) A reflection in the Y-axis.

(e) A translation k units downward.

(f) A translation h units to the left.

(g) A translation h units to the right, followed by a translation k units upward.

3. Describe the graph of each of the following functions in comparison with the graph of the function $G : x \to ax^2$, $a > 0$.

(a) $x \to a(x - h)^2$, $a > 0$, $h > 0$

(b) $x \to a(x - h)^2$, $a > 0$, $h < 0$

(c) $x \to ax^2 + k$, $a > 0$, $k > 0$

(d) $x \to ax^2 + k$, $a > 0$, $k < 0$

(e) $x \to a(x - h)^2 + k$, $a > 0$, $h > 0$, $k > 0$

(f) $x \to a(x - h)^2 + k$, $a > 0$, $h > 0$, $k < 0$

For the function $f : x \to a(x - h)^2 + h$, $a \neq 0$, we can make the following observations:

(1) The graph of f is a <u>parabola</u> with the vertical line $x = h$ as <u>axis</u> and point (h,k) as <u>vertex</u>. Figure 9 shows such a graph for the case in which $a > 0$, $h > 0$, and $k > 0$.

(2) The graph is a translation of the graph of $g : x \to ax^2$, $a \neq 0$, to the left or right $|h|$ units and upward or downward $|k|$ units.

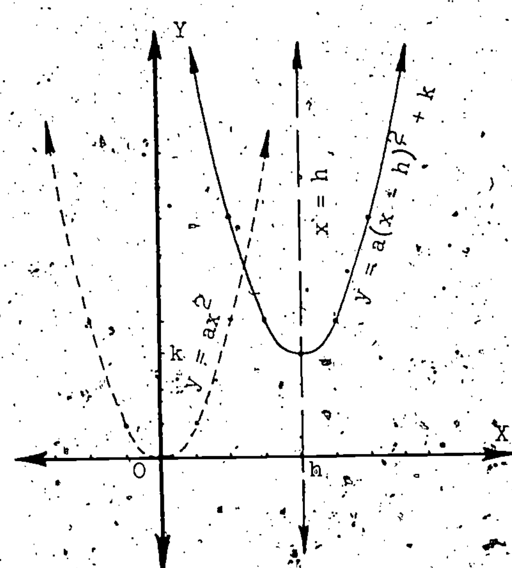


Figure 9

(3) If $a > 0$, the parabola opens upward, the vertex is its lowest point, and the function has a minimum output; if $a < 0$, the parabola opens downward, the vertex is its highest point, and the function has a maximum output.

(4) If $h = 0$, the axis is the line $x = 0$, that is, the Y-axis. If $h > 0$, the axis line $x = h$ lies to the right of the Y-axis; if $h < 0$, the axis line $x = h$ lies to the left of the Y-axis.

(5) If $k = 0$, the vertex is on the X-axis. If $k > 0$, the vertex is above the X-axis; if $k < 0$, the vertex is below the X-axis.

Exercises 24-3d

1. On one set of coordinate axes draw graphs of $f : x \rightarrow ax^2$ and $f : x \rightarrow a(x - h)^2 + k$, where $a < 0$, $h < 0$, and $k > 0$. (See Figure 9).

2. In each of the following, suppose that the graph of the given equation is subjected to the rigid motion described. Write an equation whose graph is the result of the transformation.

   (a) $y = -2x^2$; a reflection in the X-axis, followed by a translation which takes the vertex of the parabola to $(-2,0)$

   (b) $y = -\frac{1}{2}x^2 + 3$; a translation of 3 units downward, followed by a rotation of a half turn about the vertex of the parabola

   (c) $y = 3.2x^2$; a reflection in the X-axis, followed by a translation .3 units to the right and 1.5 units downward

   (d) $y = \frac{2}{3}(x - 1)^2$; a translation of $1\frac{1}{3}$ units to the left, followed by a reflection in the X-axis

   (e) $y = \frac{5}{7}x^2$; a translation of 3 units to the left, followed by a translation of $\frac{3}{7}$ units downward

   (f) $y = \frac{1}{2}x^2$; a reflection in the X-axis, followed by a translation that takes the vertex to the point $(5,2)$

   (g) $y = \frac{1}{2}x^2$; a translation that takes the vertex to the point $(5,2)$, followed by a reflection in the X-axis

   (h) $y = \frac{1}{2}x^2$; a translation that takes the vertex to the point $(5,2)$, followed by a reflection in the line $y = 2$

3. For each of the following, (1) sketch the graph of $y = 2(x - 3)^2$ and the graph of its image under the rigid motion described and (2) write an equation for the image graph.

(a) A reflection in the X-axis

(b) A reflection in the Y-axis

(c) A reflection in the X-axis followed by a reflection in the Y-axis

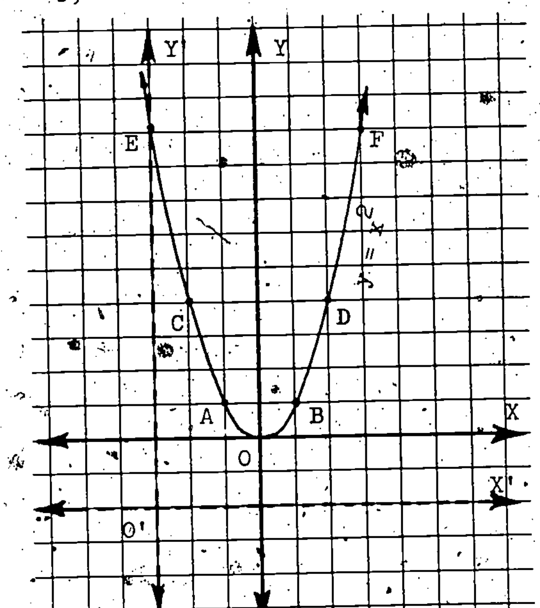
(d) A reflection in the Y-axis followed by a reflection in the X-axis.

(e) A rotation of a half turn about the origin

(f) The translation $t : (x,y) \quad (x-6,y)$

(g) A reflection in the X-axis followed by the translation $t : (x,y) \quad (x-6,y)$

(h) A reflection in the line $x = 3$, followed by a reflection in the X-axis



4. The figure shown here is the graph of the equation $y = x^2$. The graph is a parabola which opens upward. The minimum value of y is 0, the axis of symmetry is the Y-axis, the vertex is the origin.

Also indicated is a translation of the coordinate axes which takes the origin to the point $(-3,-2)$ with respect to the original axes.

(a) With respect to the X'- and Y'-axes, name the (x',y') coordinates of the following points.

O: ( )　　　D: ______

A: ______　　　E: ______

B: ______　　　F: ______

C: ______　　　(x,y): ______

(b) Since $(x', y') = (x + 3, y + 2)$, we can write

$x' = x + 3$ and $y' = y + 2$. Thus

$x = x' - 3$ and $y = $ ________.

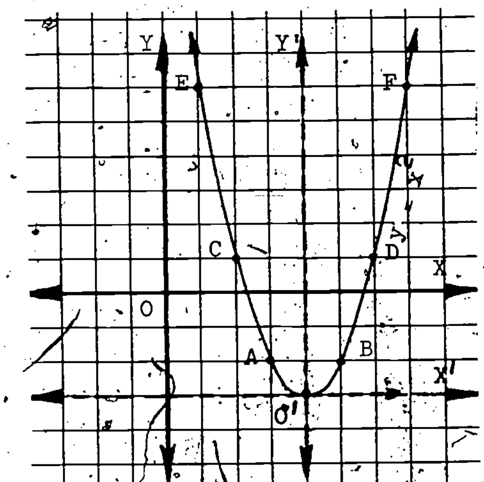
The equation $y = x^2$ becomes

$y' - 2 = $ ________, or $y' = $ ________.

The equation, $x = 0$, of the axis of the parabola becomes

________ $= 0$ or $x' = $ ________.

5. The figure shows a parabola which open upward. Its vertex is $(4, -3)$ and its axis is the line $x = 4$. The figure also indicates a translation of the coordinate axes which takes the origin $(0,0)$ to the point whose coordinates with respect to the original axes is $(4, -3)$.



(a) Name both the (x,y) and the (x',y') coordinates of the following points.

	(x,y) coordinates	(x',y') coordinates
O' :	________	________
A :	________	________
B :	________	________
C :	________	________
D :	________	________
E :	________	________
F :	________	________

(b) What is an equation of the parabola with respect to the X'- and Y'-axes?

(c) In each ordered pair, $x' = x - 4$ and $y' =$ _______; hence the equation $y' = (x')^2$ can be written $y + 2 =$ _______, or $y =$ _______.

24-4. Completing the Square

We have shown that the graph of any quadratic function expressed in the "standard" form

$$f : x \rightarrow a(x - h)^2 + k, \quad a \neq 0$$

is a parabola which we can sketch easily as a rigid motion (or a succession of rigid motions) of the graph of

$$g : x \rightarrow ax^2, \quad a > 0.$$

We made the statement at the beginning of this chapter that the graph of a function of the type $x \rightarrow ax^2 + bx + c$, $a \neq 0$ is a parabola. Now we shall explore how an expression of the form $ax^2 + bx + c$ can be rewritten in the "standard" form.

Example 1. Given the function

$$x \rightarrow x^2 + 6x,$$

how can we express it in standard form?

A geometrical interpretation of the situation can be of help. We start with a square of side x units, as shown in Figure 10. The area of the square region is x^2 square units.

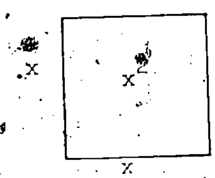


Figure 10

Now we can think of the quantity $6x$ as the sum of the areas of two rectangular regions, where each rectangle is x units long and 3 units wide. (Figure 11) The total area of the region pictured is $x^2 + 6x$ square units.

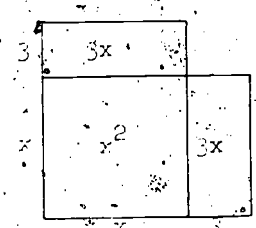


Figure 11

We <u>complete the square</u>, thus obtaining a new square region whose sides have length $x + 3$ units, and whose area is $x^2 + 6x + 9$ square units. (Figure 12)

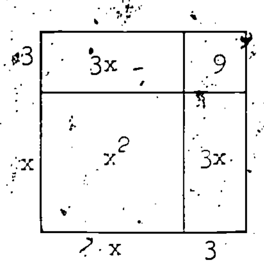


Figure 12

The area of the region in Figure 11 is nine square units <u>less</u> than the area of the large square region in Figure 12, so we can write

$$x^2 + 6x = (x^2 + 6x + 9) - 9.$$

The area of the large square region can also be expressed as $(x + 3)^2$, and we write

$$x^2 + 6x = (x + 3)^2 - 9.$$

Thus the function

$$x \to x^2 + 6x$$

can be written in standard form as

$$x \to [(x - (-3))]^2 + (-9).$$

From the standard form we see that the graph of the function has its vertex at the point $(-3, -9)$, is symmetrical to the line whose function is $x = -3$ and opens upward. It is the graph of $x \to x^2$ under a translation which shifts it 3 units to the left and 9 units down. The minimum value of the function is -9.

<u>Exercises 4-4d</u>

(Class Discussion)

1. (a) Complete:

$$x^2 - 2x - 3 = (x^2 - 2x + 1) - 3 - \underline{\qquad}$$
$$= (\underline{\qquad})^2 - \underline{\qquad}$$

(b) The function

$$f : x \to x^2 - 2x - 3$$

written in standard form is

$$f : x \to \underline{\qquad\qquad}$$

(c) Which does f have, a maximum or a minimum value? What is that value?

(d) Describe how to obtain the graph of $y = (x - 1)^2 - 4$ from the graph of $y = x^2$.

(e) Sketch the graph of

$$f : x \to x^2 - 2x - 3.$$

2. (a) Complete:

$$\begin{aligned}(p + q)^2 &= (p + q)(p + q) \\ &= (p + q)\ p + (p + q)\ \_\_\_\_ \\ &= p^2 + \_\_\_\_ + pq + \_\_\_\_ \\ &= p^2 + \_\_\_\_\ pq + q^2\end{aligned}$$

(b) Since, for any real numbers p and q, the expression $p^2 + 2pq + q^2$ can be written as the product of two identical factors, it is called a perfect square. The expression $p^2 - 2pq + q^2$ is also a perfect square and can be written as $(\_\_\_\_\_\_)^2$.

3. Fill in the missing term to make each expression a perfect square, then express it in the form $(p + q)^2$.

(a) $x^2 - \_\_\_\_\ (3x) + 9 = (\_\_\_\_\_\_)^2$

(b) $t^2 + 4t + \_\_\_\_ = (\_\_\_\_\_\_)^2$

(c) $t^2 - 4t + \_\_\_\_ = \_\_\_\_$

(d) $x^2 + 2(4x) + \_\_\_\_ = \_\_\_\_$

(e) $x^2 + 2(\quad) + 16 = \_\_\_\_$

Write each of the following in the form $(m + n)^2$.

(a) $t^2 - 3t + (\frac{3}{2})^2$

(b) $t^2 + 15t + (\frac{15}{2})^2$

(c) $x^2 + 7x + (\frac{7}{2})^2$

(d) $t^2 + 8t + 16$

(e) $x^2 - 16x + 64$

(f) $t^2 + \frac{3}{5}t + \frac{9}{100}$

(g) $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$, $a \neq 0$

Example 2. Express the function

$$g : t \to t^2 - 8t + 6$$

in standard form.

Since

$$t^2 - 8t = t^2 - 2(4)t,$$

we can write

$$t^2 - 2(4)t + 16 = (t - 4)^2.$$

Hence

$$t^2 - 8t + 6 = t^2 - 8t + 16 - 16 + 6$$
$$= (t - 4)^2 - 10.$$

Function g in standard form is

$$g : t \to (t - 4)^2 + (-10).$$

Exercises 24-4b

1. Write each of the following functions in the standard form $t \to (t - h)^2 + k$.

(a) $t \to t^2 - 2t - 3$

(b) $t \to t^2 + 4t + 4$

(c) $t \to t^2 + 4t$

(d) $t \to t^2 - 4t$

(e) $t \to t^2 + 4t + 1$

(f) $t \to t^2 - 4t - 1$

2. Write each of the following equations in the standard form $y = (x - h)^2 + k$.

(a) $y = x^2 + 2x$

(b) $y = x^2 - 2x + 3$

(c) $y = x^2 + 6x$

(d) $y = x^2 + 8x + 8$

(e) $y = x^2 - 5x$

(f) $y = x^2 + 5x + 1$

(g) $y = x^2 + \frac{5}{3} x$

(h) $y = x^2 + \frac{5}{3} x + 1$

(i) $y = x^2 + 2bx$

(j) $y = x^2 + bx$

(k) $y = x^2 + \frac{b}{a} x, \quad a \neq 0$

(l) $y = x^2 + \frac{b}{a} x + c, \quad a \neq 0$

3. Name the coordinates of the vertex of the graph of each equation in Exercise 2 above.

4. We shall wish to express functions of the type

$$t \to at^2 + bt + c$$

in the form,

$$t \to a(t - h)^2 + k,$$

where a has the same nonzero value in both forms.

(a) For the function $t \to 64t - 16t^2$, what is the value of a?

(b) For all values of t,

$$-16(t - 2)^2 + 64 = at^2 + bt + c.$$

Find values of a, b, and c which make this statement true.

So far, in practicing the method of <u>completing the square</u> in order to change a quadratic function to standard form, we have considered expressions in the form $at^2 + bt + c$ for cases in which $a = 1$. Now we shall consider cases for which a is different from 1.

<u>Example 3</u>: Write the function

$$f : x \to 3x^2 + x - 4$$

in standard form.

$$\begin{aligned} 3x^2 + x - 4 &= 3(x^2 + \tfrac{1}{3}x - \tfrac{4}{3}) \\ &= 3(x^2 + \tfrac{1}{3}x + \tfrac{1}{36} - \tfrac{1}{36} - \tfrac{4}{3}) \\ &= 3[(x + \tfrac{1}{6})^2 - \tfrac{49}{36}] \\ &= 3(x + \tfrac{1}{6})^2 - \tfrac{49}{12} \end{aligned}$$

Thus in standard form we have

$$f : x \to 3[x - (-\tfrac{1}{6})]^2 + (-\tfrac{49}{12}).$$

The vertex of the parabola is $(-\frac{1}{6}, -\frac{49}{12})$.

Exercises 24-4c

(Class Discussion)

Write each of the following functions in standard form and name the coordinates of the vertex of the graph of the function.

1. $x \to 2x^2 - 4x + 5$
2. $x \to 3x^2 - 12x - 40$
3. $x \to \frac{1}{2} x^2 + 4x - 5$
4. $x \to -x^2 + 6x + 9$
5. $x \to -2x^2 - x$
6. $x \to 3x^2 - 5x - 2$

Exercises 24-4d

1. Write each function in standard form, then state the maximum (or minimum) output value of the function and sketch its graph.

   (a) $x \to 7 + 4x - x^2$

   (b) $x \to x^2 - 3x + 4$

   (c) $x \to 3x - 2x^2$

   (d) $x \to 2x^2 - 3$

   (e) $x \to 2x^2 - 5x + 2$

2. Consider the function $f : x \to ax^2 + bx + c$, where $a > 0$. Complete these statements:

   (a) The graph of f is a ________ which opens ________ (upward, downward)

   (b) The vertex of the graph is a ________ (maximum, minimum)

   (c) The abscissa of the vertex point is ________.

   (d) $f(-\frac{b}{2a})$ ________.

   (e) The ordinate of the vertex point is ________.

3. (a) Show that $x^2 + 6x + 5 = (x + 3)^2 - 4$.

(b) Show that $(x + 3)^2 - 4 = (x + 5)(x + 1)$.

(c) Verify, using the distributive property, that

$$(x + 5)(x + 1) = x^2 + 6x + 5.$$

4. In a manner suggested by Exercise 3 above, write each of the following sums as an indicated product.

(a) $x^2 - 2x - 15$

(b) $12 - 8k + k^2$

(c) $x^2 + 11x + 18$

(d) $n^2 - 3n - 4$

(e) $6 - x - x^2$

24-5. Solving Quadratic Equations

We have seen that a function of the type

$$f : x \to ax^2 + bx + c, \quad a \neq 0,$$

is called a quadratic function. Its graph is a parabola whose axis is either the Y-axis or a line parallel to the Y-axis. We developed a procedure for changing the form of the function $f : x \to ax^2 + bx + c$, $a \neq 0$, into the form $f : x \to a(x - h)^2 + k$, $a \neq 0$. When the output is expressed as

$$a(x - h)^2 + k, \quad a \neq 0,$$

we can recognize that:

(1) the axis of symmetry of the graph is the line $x = h$;

(2) the vertex of the graph is the point (h,k);

(3) the minimum (or maximum) value of the function is k.

In discussing a function, we are often interested in knowing what inputs, if any, will result in output values of 0. Such input values are called the zeros of the function.

Exercises 24-5a

(Class Discussion)

1. Name the zero or zeros of each of the following functions.

(a) $x \to x$

(b) $x \to 2x - 3$

(c) $x \to x^2 + 3$

(d) $x \to x^2$

(e) $x \to x^2 - 4$

(f) $x \to x^2 - 2x$

(g) $x \to (x + 2)(x - 3)$

(h) $x \to x^2 + 5x + 6$

2. State the solution set of each of the following equations.

(a) $x = 0$

(b) $2x - 3 = 0$

(c) $x^2 + 3 = 0$

(d) $x^2 = 0$

(e) $x^2 - 4 = 0$

(f) $x^2 - 2x = 0$

(g) $(x + 2)(x - 3) = 0$

(h) $x^2 + 5x + 6 = 0$

3. On the basis of your answers to Exercises 1 and 2 above, write a statement about the zeros of a function in terms of the solutions of an equation.

We shall call the function

$$f : x \to ax^2 + bx + c, \quad a \neq 0,$$

the general quadratic function. The statement that the output of f, for some values of the variable, is zero,

$$ax^2 + bx + c = 0$$

we call the general quadratic equation.

Thus the zeros of the general quadratic function are the solutions of the general quadratic equation. Another word that is frequently used instead of "solution" is "root". In other words, the zeros of a function are the

roots, or solutions, of the related equation which states that the output of the function is zero.

Example 1. Find the zeros of the function $f : x \to 5x^2 - 20$.

The zeros are the roots of the equation $5x^2 - 20 = 0$.

$$5x^2 - 20 = 0 \iff 5x^2 = 20$$
$$\iff x^2 = 4$$

From our understanding of square root and of absolute value, it is clear that

$$x^2 = 4 \iff |x| = \sqrt{4}$$
$$\iff x = 2 \text{ or } x = -2$$

Thus the zeros of function f are 2 and -2.

Exercises 24-5b

1. Find the roots of each equation.

(a) $3x^2 - 27 = 0$

(b) $5(x + 2)^2 = 0$

(c) $5x^2 + 20 = 0$

(d) $5x^2 - 15 = 0$

2. Find the zeros of each function.

(a) $f : x \to 2x^2 - 2$

(b) $g : x \to 2(x - 2)^2$

(c) $h : x \to 2x^2 + 2$

3. (a) On one set of axes, sketch the graphs of the three functions in Exercise 2.

(b) Complete the following table.

function	number of zeros	number of intersections of graph with X-axis
$f : x \to 2x^2 - 2$		
$g : x \to 2(x - 2)^2$		
$h : x \to 2x^2 + 2$		

Example 2: Solve the quadratic equation

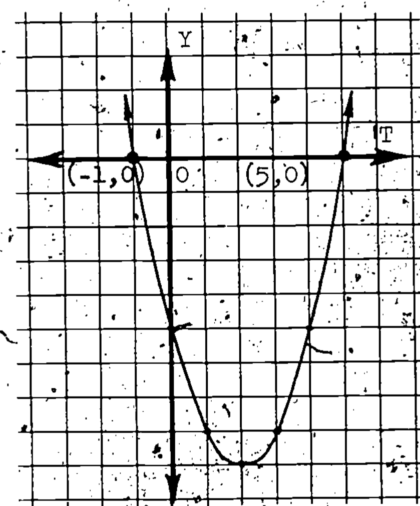
$$-16t^2 + 64t + 80 = 0$$

$$-16t^2 + 64t + 80 = 0 \Longleftrightarrow t^2 - 4t - 5 = 0.$$

Method 1. Write the function

$$f : t \to t^2 - 4t - 5,$$

in standard form, draw the graph, and find its zeros.



$t^2 - 4t - 5 = t^2 - 4t + 4 - 5 - 4$

$= (t - 2)^2 - 9.$

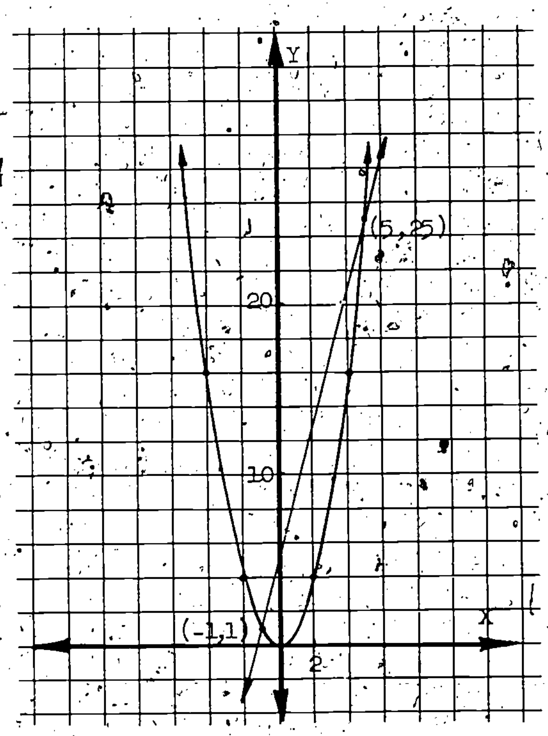
The zeros of the function are -1 and 5.

Check that -1 and 5 are roots of the equation $-16t^2 + 64t + 80 = 0$.

Method 2. Since

$$t^2 - 4t - 5 = 0 \Longleftrightarrow t^2 = 4t + 5,$$

draw the graphs of $y = t^2$ and $y = 4t + 5$ on the same set of axes.



The points of intersection are $(-1,1)$ and $(5,25)$

The abscissas of the points are roots of the equation $t^2 - 4t - 5 = 0$. Thus the solution set of $-16t^2 + 64t + 80 = 0$ is $\{-1,5\}$.

Method 3. Since

$$t^2 - 4t - 5 = (t - 2)^2 - 9,$$

we can write the following chain of equivalent sentences:

$$\begin{aligned} t^2 - 4t - 5 = 0 &\iff (t - 2)^2 - 9 = 0 \\ &\iff (t - 2)^2 = 9 \\ &\iff t - 2 = 3 \text{ or } t - 2 = -3 \\ &\iff t = 5 \text{ or } t = -1 \end{aligned}$$

The roots are 5 and -1.

Example 3: Find the roots of the equation

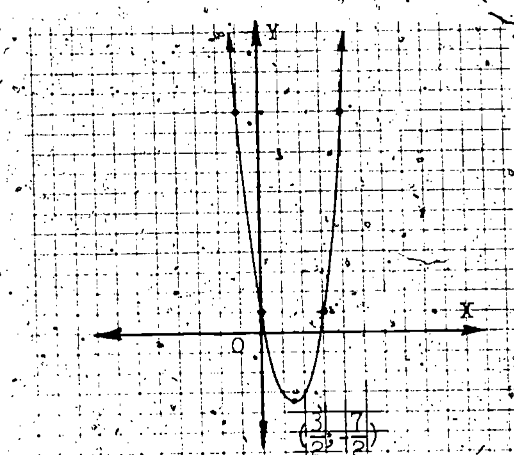
$$2x^2 - 6x + 1 = 0.$$

$$\begin{aligned} 2x^2 - 6x + 1 = 0 &\iff 2(x^2 - 3x + \tfrac{1}{2}) = 0 \\ &\iff 2(x^2 - 3x + \tfrac{9}{4} - \tfrac{9}{4} + \tfrac{1}{2}) = 0 \\ &\iff 2[(x - \tfrac{3}{2})^2 - \tfrac{7}{4}] = 0 \\ &\iff 2(x - \tfrac{3}{2})^2 - \tfrac{7}{2} = 0 \end{aligned}$$

From the equation in standard form we can sketch the graph of the function

$$f : x \to 2x^2 - 6x + 1$$

and observe that the zeros of the function are not integers. It appears that one of the zeros is a number between 0 [illegible] the other zero is between 2 and 3.



However, we can get more information about the roots of the equation $2x^2 - 6x + 1 = 0$ if we use Method 3, which we shall refer to as "Solution by Completing the Square".

Since $\quad 2x^2 - 6x + 1 = 0 \iff 2[(x - \frac{3}{2})^2 - \frac{7}{4}] = 0$

we write $\quad 2x^2 - 6x + 1 = 0 \iff (x - \frac{3}{2})^2 - \frac{7}{4} = 0$

$$\iff (x - \tfrac{3}{2})^2 = \tfrac{7}{4}$$

$$\iff x - \tfrac{3}{2} = \tfrac{\sqrt{7}}{2} \text{ or } x - \tfrac{3}{2} = -\tfrac{\sqrt{7}}{2}$$

$$\iff x = \tfrac{3}{2} + \tfrac{\sqrt{7}}{2} \text{ or } x = \tfrac{3}{2} - \tfrac{\sqrt{7}}{2}.$$

Thus the roots are $\frac{3 + \sqrt{7}}{2}$ and $\frac{3 - \sqrt{7}}{2}$.

Since $\sqrt{7} \approx 2.646$, we can compute 3-decimal place approximations to the roots as follows:

$$\frac{3 + \sqrt{7}}{2} \approx \frac{3 + 2.646}{2} = \frac{5.646}{2} = 2.823$$

$$\frac{3 - \sqrt{7}}{2} \approx \frac{3 - 2.646}{2} = \frac{0.354}{2} = 0.177$$

Compare these estimates with what the graph shows about the zeros of the function

$$f : x \to 2x^2 - 6x + 1.$$

Exercises 24-5c

1. Solve each of the following equations by completing the square. For any irrational roots, give approximations to 3 decimal places.

   (a) $5 - 4x - x^2 = 0$
   (b) $x^2 - 4x + 5 = 0$
   (c) $\frac{1}{4}x^2 + 2x + 4 = 0$
   (d) $x^2 - 8x + 14 = 0$
   (e) $x^2 + 2x - 8 = 0$
   (f) $x^2 + 2x + 8 = 0$
   (g) $5x = 1 + 2x^2$
   (h) $4x^2 + 1 = 4x$

2. (a) Sketch the graph of each of the following functions.

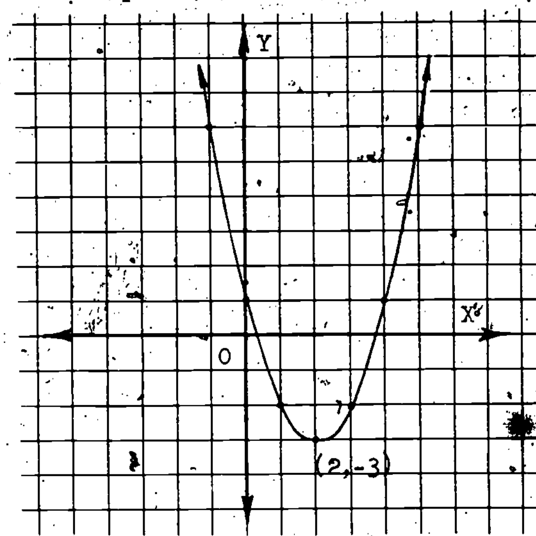
   (1) $f : x \to x^2 - 2x - 1$
   (2) $g : x \to 2x^2 - 2x - 5$
   (3) $h : x \to x^2 + 2x + 1$
   (4) $j : x \to -2x^2 - 2x - 5$

   (b) Solve each of the following equations, giving 3-decimal approximations for irrational roots.

   (1) $x^2 - 1 = 2x$
   (2) $2x^2 - 5 = 2x$
   (3) $x^2 + 1 = -2x$
   (4) $-2x^2 - 5 = 2x$

(c) Explain how the graphs in part (a) can help you to check on the reasonableness of the solutions in part (b).

3. (a) Write in standard form the function f whose graph is shown.



(b) Write function f in general quadratic form.

(c) Find the zeros of function f to 3 decimal places.

(d) Use the graph to find the values of x for which:

(1) $f(x) = -3$ (Hint: What is the intersection of the parabola with the line $y = -3$?)

(2) $f(x) = -2$

(3) $f(x) = 1$

(4) $f(x) = 6$

(e) Find the roots of each of the following equations.

(1) $x^2 - 4x + 4 = 0$

(2) $x^2 - 4x + 3 = 0$

(3) $x^2 - 4x + 1 = 0$

(4) $x^2 - 4x = 0$

(5) $x^2 - 4x - 5 = 0$

4. Find the dimensions of the rectangle with perimeter 72 feet which encloses a region of maximum area.

(a) If x represents a number of feet in the width of a rectangle whose perimeter is 72 feet, write a description of the output of each of the following functions.

$f : x \to 2x$

$g : x \to 72 - 2x$

$h : x \to \frac{1}{2}(72 - 2x)$

$F : x \to x(36 - x)$

(b) To answer the problem, we need to determine the value of x for which the output $F(x)$ is a maximum. Write function F in standard form and describe its graph.

(c) What point is the vertex of the parabola? What is the maximum output of function F? For what value of x is $F(x)$ a maximum? What is the corresponding value of $36 - x$?

(d) What are the required dimensions of the rectangle? What is its area?

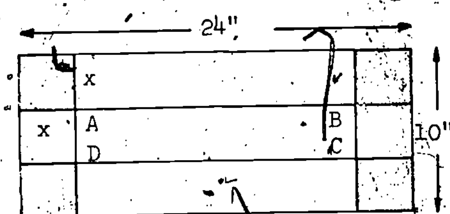
5. A rectangular field is to be fenced on three sides, with a river serving as a natural fence on the fourth side. If 100 yards of fencing is available, find the dimensions of the field with largest area.

6. (a) Write the function $f : x \rightarrow x - x^2$ in standard form.

(b) Find the number which most exceeds its square.

7. Recall the discussion of the "Golden Rectangle", in which the ratio of the width to the length is equal to the ratio of the length to the sum of the length and the width. With these pleasing proportions in mind, design a rectangular picture frame having a perimeter of 60 inches. Give the dimensions to the nearest tenth of an inch.

8. From a rectantular sheet of aluminum 24 inches long and 10 inches wide, an open box is made by cutting squares of equal area from the four corners and folding up the edges. The area of the base of the box is 72 square inches.



(a) What is the area of each corner square?

(b) What is the volume of the box?

9. Among all pairs of numbers whose sum is 50, determine the pair whose product is greatest.

10. (a) Write the function $f : x \rightarrow x^2 - 6x + 10$ in standard form.

(b) What is the minimum value of f?

(c) Find the maximum value of the function

$$g : x \rightarrow \frac{2}{x^2 - 6x + 10}.$$

(d) For what value of x is the output value of g a maximum?

____

24-6. "Falling Body" Functions

In the physical world, some of the most interesting instances in which quadratic functions are used to model physical situations are applications of Galileo's principle about falling objects. You will recall that Galileo showed that the distance traveled by a falling body is a function of the time spent in falling.

When t represents a number of seconds, the output of the function

$$f : t \rightarrow 16t^2, \quad t \geq 0$$

is the number of feet the object falls in t seconds. This, of course, applies to an object dropped from a height. In this section we shall see how the principle applies also to objects which are projected upward instead of merely being dropped, and to associating the time with the height the object is above the ground at any instant.

Example 1. A ball is dropped from a 47th story window of the Time-Life Building in New York City. If the height of the window above the street is 576 feet, write a quadratic function whose output is the height (in feet) of the ball above the ground at any given second after it is released.

If t represents a number of seconds, then

Function	Description of Output
$f : t \rightarrow 576$	Height in feet of ball above the street at instant of release
$g : t \rightarrow 16t^2, \quad t \geq 0$	Number of feet the ball falls in t seconds
$F : t \rightarrow 576 - 16t^2, \quad t \geq 0$	Height in feet of the ball at the end of t seconds

Thus $F : t \rightarrow 576 - 16t^2$, $t \geq 0$ is the required function.

Exercises 24-6a

1. Use the function $F : t \rightarrow 576 - 16t^2$, $t \geq 0$ to answer the following questions about the situation described in Example 1.

   (a) How many feet above the ground is the ball at the end of 2 seconds?

   (b) How many seconds would it take for a ball dropped from a 47th story window of the Time-Life Building to hit the pavement below?

2. Horatio Lox stood at his office window on the 37th floor of the Time-Life Building. His cousin, Horatio Algae stood beside the window of his publishing office on the 47th floor directly above the office of Lox. At a prearranged signal, Algae folded his newspaper and dropped it out of his window, into the waiting hands of Lox. If it took exactly 3 seconds for the paper to get from Algae to Lox, how high above the ground is a 37th floor window?

3. A ball is dropped from the top of the Fidelity Union Tower in Dallas, Texas. After t seconds the height s feet of the ball above the ground is given by

$$s = 400 - 16t^2 \text{ and } t \geq 0.$$

   (a) What is the height of Fidelity Union Tower?

   (b) How long does it take for the ball to reach the ground?

4. The Woolworth Building in New York City is about 784 feet high. A ball is dropped from the top of the Woolworth Building.

   (a) Write a function whose output is the height in feet of the ball above the ground at the end of t seconds.

   (b) How long does it take for the ball to reach the ground?

5. A flowerpot falls from a 75th story windowsill of the Chrysler Building in New York City. We know that after t seconds the height s feet of the flowerpot above the ground is given by the equation

$$s = 1024 - 16t^2 \text{ and } t \geq 0.$$

   (a) How long does it take for the flowerpot to hit the sidewalk at the corner of Lexington Avenue and Forty-Second Street directly beneath the window?

(b) The distance from the 75th story windowsill of the Chrysler Building to the roof of the building is 22 feet. How tall is the Chrysler Building?

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Example 2. A ball is thrown from ground level straight up with an initial speed of 64 ft/sec. Since it has an initial upward speed of 64 ft/sec, and also is acted upon by the force of gravity, at some instant it will stop rising and begin to fall. Write a function whose output is the height in feet of the ball above the ground at any second, t.

If t represents a number of seconds, then

Function	Description of Output
$f : t \to 64t, \quad t \geq 0$	Number of feet traveled in t seconds at 64 ft/sec
$g : t \to 16t^2, \quad t \geq 0$	Number of feet an object falls in t seconds
$F : t \to 64t - 16t^2, \quad t \geq 0$	Height in feet of ball above the ground after t seconds

Exercises 24-6b

1. Use the function $F : t \to 64t - 16t^2, \ t \geq 0$ to answer the following questions about the situation described in Example 2.

(a) Write the function in standard form.

(b) Does the function have a maximum or a minimum value? What is that value?

(c) What is the greatest height reached by the ball? In how many seconds does it reach that height?

(d) What are the zeros of the function? In how many seconds does the ball return to the ground?

(e) In how many seconds is the height of the ball above the ground 48 feet?

2. A ball is thrown straight up from the ground with an initial speed of 48 ft/sec. Write a function whose output is the height in feet

of the ball above the ground in t seconds, and use it to answer the following questions.

(a) After how many seconds does the ball reach its highest point?

(b) How high does the ball go before it begins to fall?

(c) As a mathematical model of the physical situation, what is the domain of the function?

3. A pellet is projected straight up. After a while it comes down, in the same vertical path, to the place on the ground from which it was launched. After t seconds the height s feet of the pellet above the ground is described by the equation

$$s = 160t - 16t^2.$$

(a) After how many seconds does the pellet reach its maximum height?

(b) How high does the pellet go?

(c) What was the initial speed with which the pellet was projected?

(d) After how many seconds does the pellet return to the ground?

(e) For what set of values of t does the equation $s = 160t - 16t^2$ actually serve as a mathematical model of the physical situation?

Example 3. Suppose that you stand close to the edge on the top of a building 80 feet tall. You throw a ball upward with an initial speed of 64 ft/sec in a nearly vertical path. As it descends the ball just misses the edge of the building and lands on the ground below. Write a function whose output is the height in feet of the ball above the ground at the end of t seconds.

If t represents a number of seconds, then

Function	Description of Output
$f : t \to 80$	Initial height of the ball above the ground
$g : t \to 64t, \quad t \geq 0$	Number of feet traveled in t seconds at 64 ft/sec
$h : t \to 16t^2, \quad t \geq 0$	Number of feet an object falls in t seconds
$F : t \to 80 + 64t - 16t^2, \quad t \geq 0$	Height in feet of ball above the ground after t seconds

Exercises 24-6c

1. Use the function $F: t \to 80 + 64t - 16t^2$, $t \geq 0$ to answer the following questions about the situation described in Example 3.

(a) Find $F(0)$: What is the height of the ball above the ground after zero seconds?

(b) Find $F(1)$ and $F(3)$. What is the height of the ball above the ground after 1 second? After 3 seconds?

(c) Find the values of t for which the output of F is 80. How many seconds after the ball is thrown does it pass the edge of the top of the building as it falls to the ground?

(d) What is the maximum height of the ball above the ground? After how many seconds does it reach its maximum height?

(e) After how many seconds does the ball reach the ground?

(f) For what set of values of t does the function F serve as a mathematical model?

2. From the top of a tower 160 feet tall a ball is thrown downward with an initial speed of 48 ft/sec.

(a) Write a function whose output is the height in feet of the ball above the ground at the end of t seconds.

(b) Find the zeros of the function. In how many seconds does the ball reach the ground?

(c) As a mathematical model of the physical situation, what is the domain of the function?

3. A railroad flatcar moves through a station on a track lowered so that the platform of the flatcar is on the same level as the station platform. The train is moving at a speed of 32 ft/sec. Tom stands on the car; as he passes Dick on the station platform, Tom throws a ball straight upward with an initial speed of 64 ft/sec. After t seconds the ball is at a horizontal distance of x feet and a vertical distance of y feet from the point where it was thrown. The distances x feet and y feet are given by the equations

$$x = 32t$$

and

$$y = 64t - 16t^2.$$

(a) Why does Tom not have to move to catch the ball?

(b) What is the path of the ball as Dick sees it from the platform?

(c) Write y in terms of x. [Hint: $t = \underline{\ ?\ } x$]

(d) Describe the graph of $y = 2x - \frac{x^2}{64}$.

(e) For what values of x does $y = 0$?

(f) What is t when $x = 128$?

(g) After how many seconds does Tom catch the ball?

(h) How far down the platform from Dick does Tom catch the ball?

24-7. The Use of Quadratics in Solving Other Equations

We have seen how to solve quadratic equations of the general form $ax^2 + bx + c = 0$, for particular values of a, b and c, $a \neq 0$. We can also deal with other equations which are not quadratic in form, but for which it is helpful to write an equivalent open sentence which involves a quadratic. One type of equation for which this is true is an equation which involves a square root.

Example 1. Consider the sentence

$$\sqrt{x} = 2 - x.$$

For this sentence to be meaningful, we must state certain restrictions. (1) We know that if $\sqrt{x}$ is a real number, then x is non-negative; that is, $x \geq 0$. (2) By definition, $\sqrt{x}$ represents the non-negative number whose square is x, that is, $2 - x \geq 0$.

$$2 - x \geq 0 \iff -x \geq -2$$
$$\iff x \leq 2$$

Thus we write the compound sentence $\sqrt{x} = 2 - x$ and $0 \leq x \leq 2$. In order to solve this sentence we need to square both sides of $\sqrt{x} = 2 - x$.

Squaring both sides of an equation does not always produce an equivalent equation. For real numbers a and b, if $a = b$, then obviously $a^2 = b^2$. However, if $a^2 = b^2$, can we be sure

that $a = b$? No, since if $a = -b$ it is also true that $a^2 = b^2$. However, the compound sentence

$$a = b \text{ and } a \geq 0, \ b \geq 0$$

is equivalent to $a^2 = b^2$ and $a \geq 0$, $b \geq 0$.

Thus we can write

$$\sqrt{x} = 2 - x \text{ and } 0 \leq x \leq 2$$

$$\Longleftrightarrow x = (2 - x)^2 \text{ and } 0 \leq x \leq 2$$

$$\Longleftrightarrow x = 4 - 4x + x^2 \text{ and } 0 \leq x \leq 2$$

$$\Longleftrightarrow x^2 - 5x + 4 = 0 \text{ and } 0 \leq x \leq 2$$

Solving $x^2 - 5x + 4 = 0$ by the methods of the previous section, we find that an equivalent sentence is

$$(x = 1 \text{ or } x = 4) \text{ and } 0 \leq x \leq 2.$$

Hence the solution set of $\sqrt{x} = 2 - x$ is $\{1\}$.

Exercises 24-7a

(Class Discussion)

For each of the following equations, write a compound sentence indicating the restrictions on the variable.

1. $\sqrt{2x} = 1 + x$
2. $\sqrt{2x + 1} = x + 1$
3. $\sqrt{x + 1} - 1 = x$
4. $\sqrt{4x} - x + 3 = 0$
5. $3\sqrt{x + 13} = x + 9$
6. $\sqrt{5x - 1} = 1 - \sqrt{x}$
7. $\sqrt{4x - 3} = \sqrt{8x + 1} - 2$

Exercises 24-7b

State the solution set of

1. $\sqrt{2x} = 1 + x$ and $x \geq 0$
2. $\sqrt{2x + 1} = x + 1$ and $x \geq -\frac{1}{2}$
3. $\sqrt{x + 1} - 1 = x$ and $x \geq -1$
4. $\sqrt{4x} + x - 3 = 0$ and $x \geq 3$
5. $\sqrt{4x - 3} = \sqrt{8x + 1} - 2$ and $x \geq \frac{3}{4}$

Example 2. Solve $|x| - x = 1$, which is equivalent to $|x| = x + 1$, for all members of the replacement set.

For this sentence to be meaningful, what restriction is there on x. Since $|x|$ has been defined for all real numbers, there is no reason to restrict x to non-negative numbers. However, recall from the definition of $|x|$,

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

that, no matter whether x represents a negative or a non-negative number, the absolute value of x (in symbolic form $|x|$) is always non-negative. Thus the number represented by $x + 1$ must be non-negative.

$$x + 1 \geq 0 \iff x \geq -1$$

Thus we write the compound sentence

$$|x| - x = 1 \text{ and } x \geq -1.$$

With this restriction, we can now write the following chain of equivalences:

$$\begin{aligned} |x| - x = 1 \text{ and } x \geq -1 &\iff |x| = x + 1 \text{ and } x \geq -1 \\ &\iff |x|^2 = (x + 1)^2 \text{ and } x \geq -1 \\ &\iff x^2 = x^2 + 2x + 1 \text{ and } x \geq -1 \\ &\iff 0 = 2x + 1 \text{ and } x \geq -1 \\ &\iff x = -\tfrac{1}{2} \text{ and } x \geq -1 \end{aligned}$$

The solution set is $\{-\frac{1}{2}\}$.

Exercises 24-7c

For each of the following equations, (a) write a compound sentence indicating the restrictions on the variable if there are any, and (b) state the solution set of the compound sentence.

1. $|2x| = x + 1$
2. $2x = |x| + 1$
3. $x = |2x| + 1$
4. $x - |x| = 1$
5. $|x - 2| = 3$

Example 3. Solve $1 + \frac{12}{x^2 - 4} = \frac{3}{x - 2}$.

For this sentence to be meaningful, we restrict the variable to $x \neq -2$, $x \neq 2$. Thus we write this chain of equivalences:

$$1 + \frac{12}{x^2 - 4} = \frac{3}{x - 2} \text{ and } x \neq -2 \text{ and } x \neq 2$$

$$\Longleftrightarrow x^2 - 4 + 12 = 3x + 6 \text{ and } x \neq -2, 2$$

$$\Longleftrightarrow x^2 - 3x + 2 = 0 \text{ and } x \neq -2, 2$$

$$\Longleftrightarrow (x = 2 \text{ or } x = 1) \text{ and } x \neq -2 \text{ and } x \neq 2$$

$$\Longleftrightarrow x = 1$$

The solution set is $\{1\}$.

Exercises 24-7d

For each of the following, (a) write a compound sentence indicating the restrictions on the variable, and (b) state the solution set of the compound sentence.

1. $x + \frac{1}{x} = 2$
2. $y + \frac{2}{y} = 1$
3. $\frac{1}{y} - \frac{1}{y - 4} = 1$
4. $\left(\frac{x - 1}{x + 1}\right)^2 = 4$
5. $\frac{1}{x} = 1 - \frac{6}{x + 4}$
6. $\frac{7}{x - 1} = \frac{6}{x^2 - 1} + 5$
7. $\frac{x}{x + 2} = \frac{4}{x + 1} - \frac{2}{x + 2}$

For each of the following problems: (a) analyze the situation and write an equation which is a suitable model (b) solve the equation and (c) interpret the solution and answer the question in the problem.

8. Find a positive number which when diminished by 14 is equal to 51 times the reciprocal of the number.

9. One pipe alone can fill a tub in 6 minutes less time than it takes a second pipe alone to fill the same tub. Both pipes together can fill the tub in 4 minutes. How long does it take each pipe alone to fill the tub?

10. A motorboat can average 12 miles per hour in still water. To make a round trip of 16 miles upstream and 16 miles back requires 3 hours. What is the rate of the current?

24-8. Summary

Section 24-1.

The function

$$f : x \to ax^2 + bx + c, \quad a \neq 0$$

is called a quadratic function. The graph of such a function is called a parabola.

Section 24-2.

The parabola which is shown in Figure 11 is the graph of the function

$f : x \to ax^2 + bx + c$, $a = 1$, $b = 0$, $c = 0$.

That is, it is the graph of

$f : x \to x^2$,

which we refer to as the "basic" quadratic function.

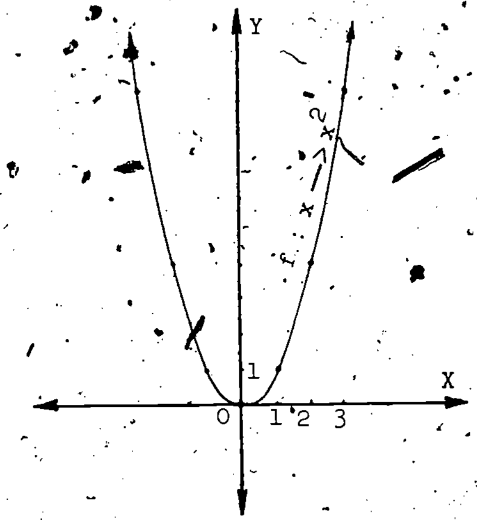

Figure 11

For $a > 0$ the graph of $f : x \to ax^2$ is a parabola that opens upward, with its lowest point at the origin. Its domain is the set of all real numbers, its range is $\{y : y \geq 0\}$. Except at the origin, the graph of $f : x \to ax^2$ lies below the graph of the basic function if $0 < a < 1$: it lies above the graph of the basic function if $a > 1$.

For $a > 0$, the graph of $g : x \to -ax^2$ is a parabola that opens downward, with its highest point at the origin. The graph is the reflection on the X-axis of the graph of $f : x \to ax^2$. The domain of function g is the set of all real numbers; the range of function g is $\{y : y \leq 0\}$.

A frequently used notation for indicating the output of a function f for a given input n is $f(n)$, read "f of n", or "f at n". Thus

$$f : x \to f(x), \quad \text{where } f(x) = x^2$$

says "function f assigns to each input x from the domain an output $f(x)$ such that $f(x)$ is x^2."

The graph of $f : x \to x^2$ is also referred to as the graph of $f(x)$, and as the graph of $y = x^2$. Any point on the graph can be described as (x,y), or as (x,x^2), or as $(x,f(x))$.

Section 24-3.

The graph of the function

$$f : x \to ax^2 + k, \quad a \neq 0$$

is a translation of the graph of $g : x \to ax^2$, $a \neq 0$. If $k > 0$, the translation is upward a distance of k units; if $k < 0$, the translation is downward a distance of $|k|$ units.

The graph of

$$f : x \to a(x - h)^2, \quad a \neq 0$$

is also a translation of the graph of $g : x \to ax^2$, $a \neq 0$. This time the translation is to the right a distance of h units if $h > 0$, and to the left $|h|$ units if $h < 0$.

The graph of

$$f : x \to a(x - h)^2 + k, \quad a \neq 0$$

is a translation of the graph of $g : x \to ax^2$, $a \neq 0$, to the left or right $|h|$ units and upward or downward $|k|$ units.

Each parabola has an axis of symmetry, referred to as the axis of the parabola. The intersection of a parabola with its axis is called the vertex of the parabola. For the function $f : x \to a(x - h)^2 + k$, $a \neq 0$, if $a > 0$, the graph opens upward and the vertex (h,k) is its lowest point; if $a < 0$, the graph opens downward and the point (h,k) is its highest point. The value of k is the minimum (or maximum) output value of function f, and the value of h is the input for which k is the output.

Section 24-4.

The general quadratic function

$$f : x \to ax^2 + bx + c, \quad a \neq 0$$

can be rewritten in the "standard" form

$$f : x \to a(x - h)^2 + k, \quad a \neq 0$$

by a procedure called "completing the square".

Section 24-5.

For any function, the inputs which result in output values of 0 are called zeros of the function. The zeros of a function are the roots (or solutions) of the equation which states that the output of the function is zero.

The set of real numbers which are roots of any quadratic equation can be determined by rewriting the expression which is the output of the function in "standard" form. Then

$$a(x - h)^2 + k = 0 \text{ and } a \neq 0 \iff a(x - h)^2 = -k \text{ and } a \neq 0$$
$$\iff (x - h)^2 = -\frac{k}{a} \text{ and } a \neq 0.$$

If $-\frac{k}{a} < 0$, the equation has no real roots; otherwise, it has either one or two real roots.

Section 24-6.

Quadratic equations can be used to model physical situations which apply Galileo's observation that the distance traveled by a falling object is a function of the time spent in falling.

If t represents a number of seconds and $t \geq 0$, then

Function	Description of Output
$f : t \to 16t^2$	Number of feet an object falls in t seconds.
$g : t \to a - 16t^2$	Distance (in feet) above ground after t seconds of an object dropped from a height of a feet
$h : t \to bt - 16t^2$	Distance (in feet) above ground after t seconds of an object tossed upward at a rate of b ft/sec
$F : t \to a + bt - 16t^2$	Distance (in feet) above ground after t seconds of an object tossed upward from a height of a feet at a rate of b ft/sec

Section 24-7.

Some equations which are not quadratic in form can be solved by methods which lead to quadratic equations.

(1) If the equation involves a square root, squaring both sides may be helpful. Since

$$a = b \text{ and } a \geq 0, \ b \geq 0 \Longleftrightarrow a^2 = b^2 \text{ and } a \geq 0, \ b \geq 0$$

it is necessary first of all to write a compound sentence stating restrictions on the variable to insure that the number under the radical sign is non-negative, and that the square root is a positive number.

(2) Squaring both sides of an equation may also be helpful if the equation concerns the absolute value of an expression involving the variable. In this case, the restriction on the variable must insure that the absolute value is a positive number.

(3) Another situation which may lead to a quadratic equation is the solution of a fractional equation. With restrictions stated to insure that no denominator has the value 0, applying the multiplication property of equality may lead to a quadratic equation.